Homework 7: Exploration of Velocity Ratios

a) For equal core and bypass velocities, according to the notes, we must have:

$$\tau_f = \frac{1+\theta_t+(1+\alpha-\tau_c)-\frac{\theta_t}{\theta_0\tau_c}}{\theta_0(1+\alpha)} \qquad (1)$$

For maximum thrust (maximum τ_f as well), we choose:

$$\tau_c = \frac{\sqrt{\theta_t}}{\theta_0} \qquad (2)$$

$\theta_t = 6.25$

$\theta_0 = 1 + 0.2 * 0.9^2 = 1.162$

$\tau_c = 2.1515$

Using $\alpha = 6$, we calculate:

$$\tau_f = \frac{1+6.25+1.162(7-2.1515)-\frac{6.25}{1.162*2.1515}}{1.162*7} = 1.2766$$

$$\pi_f = \tau_f^{\frac{\gamma}{\gamma-1}} = 1.2766^{3.5} = 2.3508$$

This is more than the imposed $\pi_{f,max} = 2$.

For this hypothetical design, we would have for the turbine:

$$\tau_t = 1 - \frac{\theta_0}{\theta_t}\left[\tau_c - 1 + \alpha(\tau_f - 1)\right] \qquad (3)$$

$$\tau_t = 1 - \frac{1.162}{6.25}[2.1515 - 1 + 6(1.2766 - 1)] = 0.4773$$

Both jet speeds are now given by:

$$\frac{u_6}{u_0} = \frac{u_8}{u_0} = \frac{1}{M_0}\sqrt{\frac{2}{\gamma-1}\left(\theta_0\tau_f - 1\right)} \qquad (4)$$

$$\frac{u_6}{u_0} = \frac{u_8}{u_0} = \frac{1}{0.9}\sqrt{5(1.162 \times 1.2766 - 1)} = 1.7275$$

$$u_0 = M_0\sqrt{\gamma R T_0} \qquad (5)$$

$$u_0 = 0.9\sqrt{1.4 \times 287 \times 220} = 267.7 \ m/s$$

$$u_6 = u_8 = 267.7 \times 1.7275 = 462.4 \ m/s$$

The core normalized thrust would be:

$$\frac{F_{core}}{\dot{m}a_0} = \frac{\dot{m}(u_6-u_0)}{\dot{m}a_0} = M_0\left(\frac{u_6}{u_0} - 1\right) = 0.9(1.7275 - 1) = 0.6548 \qquad (6)$$

The fan would contribute α times as much:

$$\frac{F_{bp}}{\dot{m}a_0} = 6 \times 0.6548 = 3.9285$$

For the total:

$$\frac{F}{\dot{m}a_0} = 4.5833$$

The specific impulse would have been:

$$I = \frac{F}{g\dot{m}_f} = \frac{Fh}{g\dot{m}c_p(T_{t4}-T_{t3})} = \frac{a_0 h}{c_p T_0}\frac{\left(F/\dot{m}a_0\right)}{\theta_t - \theta_0 \tau_c} \qquad (7)$$

$$I = \frac{297.3\times43\times10^6}{1004.5\times220}\frac{4.583}{6.25-1.162\times2.1515} = 7213\ s$$

The overall efficiency is directly related to the specific impulse:

$$\eta_{ov} = \frac{a_0 g I}{h} = \frac{267.7\times9.8\times7213}{43\times10^6} = 0.4399 \qquad (8)$$

The propulsive efficiency follows from:

$$\eta_p = \frac{2}{\frac{u_6}{u_0}+1} = \frac{2}{\frac{u_8}{u_0}+1} \qquad (9)$$

$$\eta_p = \frac{2}{1.7275+1} = 0.7333$$

By ratio:

$$\eta_{th} = \frac{\eta_{ov}}{\eta_p} = \frac{0.4399}{0.7333} = 0.5999$$

b) We now accept instead the maximum allowable fan ratio $\pi_f = 2$.

$$\tau_f = \pi_f^{\frac{\gamma-1}{\gamma}} = 2^{\frac{\gamma-1}{\gamma}} = 1.21901$$

Keep $\theta_t = 6.25, \alpha = 6$. The bypass flow speed is now:

$$u_8 = \frac{u_0}{M_0}\sqrt{\frac{2}{\gamma-1}\left(\theta_0\tau_f-1\right)} = \frac{\sqrt{5(1.162\times1.21901-1)}}{0.9}\times 267.7 = 429.2\ m/s$$

(instead of 462.4 m/s)

The normalized thrust from the bypass flow is:

$$\frac{F_{bp}}{\dot{m}a_0} = \alpha M_0\left(\frac{u_8}{u_0}-1\right) = 6\times0.9\left(\frac{429.2}{267.7}-1\right) = 3.2584$$

(instead of 4.583)

For the contribution of the core, we have:

$$\frac{F_{core}}{\dot{m}a_0} = \sqrt{\frac{2}{\gamma-1}\frac{\theta_t}{\theta_0\tau_c}\left(\theta_0\tau_c\tau_t-1\right)} - M_0 \qquad (10)$$

$$\tau_t = 1 - \frac{\theta_0}{\theta_t}\left[\tau_c - 1 + \alpha(\tau_f - 1)\right] \qquad (11)$$

Substituting and simplifying:

$$\frac{F_{core}}{\dot{m}a_0} = \sqrt{\frac{2}{\gamma-1}\left[\theta_t - \theta_0\alpha(\tau_f - 1) - \theta_0(\tau_c - 1) - \frac{\theta_t}{\theta_0\tau_c}\right]} - M_0 \qquad (12)$$

We can see here that this part of the thrust depends on τ_c, and the optimum is when $-\tau_c - \frac{\theta_t}{\theta_0\tau_c}$ is a maximum. This leads again to $(\tau_c)_{opt} = \frac{\sqrt{\theta_t}}{\theta_0}$, _same as before_. So we keep $\tau_c = 2.1515$.

Substituting:

$$\frac{F_{core}}{\dot{m}a_0} = \sqrt{5\left[6.25 - 1.162 \times 6(1.21901 - 1) - 1.162(2.1515 - 1) - \frac{6.25}{1.162\times2.1515}\right]} - 0.9 = 1.20364$$

(instead of 0.654)

The core is now producing more thrust than before because the turbine extracts less power for the fan. In fact:

$\tau_f = 1 - \frac{1.162}{6.25}(1.1515 + 6 \times 0.21901) = 0.5416$

(instead of 0.4793)

For the total thrust:

$\frac{F}{\dot{m}a_0} = 3.2584 + 1.2036 = 4.462$

(instead of 4.5833)

There is some thrust reduction, but only 2.6%.

The core jet speed follows from:

$\frac{F_{core}}{\dot{m}a_0} = M_0\left(\frac{u_6}{u_0} - 1\right)$

$u_6 = u_0\left[1 + \frac{F_{core}/\dot{m}a_0}{M_0}\right] = 267.7\left(1 + \frac{1.2036}{0.9}\right) = 625.7 \ m/s$

(instead of 462.4 m/s)

The expression for specific impulse is as before, where only the factor $\frac{F}{\dot{m}a_0}$ has changed. We get:

$I = 7213\frac{4.462}{4.5833} = 7022 \ s$

(which is 2.6% lower)

The overall efficiency is also reduced in the same fraction:

$\eta_{ov} = 0.4399\frac{4.462}{4.5833} = 0.4283$

The propulsive efficiency is, by definition:

$$\eta_p = \frac{Fu_0}{kinetic\ energy\ increase} \qquad (13)$$

Both streams contribute to numerator and denominator:

$$\eta_p = \frac{\dot{m}u_0[u_6-u_0+\alpha(u_8-u_0)]}{\dot{m}\left[\frac{u_6^2-u_0^2}{2}+\alpha\frac{u_8^2-u_0^2}{2}\right]} = \frac{2\times267.7(625.7+6\times429.2-7\times267.7)}{625.7^2+6\times429.2^2-7\times267.7^2} = 0.714 \qquad (14)$$

(instead of 0.7333)

The thermodynamic efficiency can be found by ratio as:

$$\eta_{th} = \frac{\eta_{ov}}{\eta_p} = \frac{0.4283}{0.714} = 0.5998$$

Within truncation error, this is *the same as before*.

Concept Questions

1) Why is η_{th} unchanged? The heat-to-work conversion is done by the core, which runs as a Brayton cycle with a compression ratio which is:

$$\frac{P_{t3}}{P_0} = 1 - (\theta_0\tau_c)^{\frac{\gamma}{\gamma-1}}$$

This is *unchanged* from before. In fact, using the Brayton cycle efficiency expression:

$$\eta_{th} = 1 - \left(\frac{P_0}{P_{t3}}\right)^{\frac{\gamma-1}{\gamma}} = 1 - \frac{1}{\theta_0\tau_c}$$

Since:

$$\tau_c = \frac{\sqrt{\theta_t}}{\theta_0}$$

Thermodynamic efficiency becomes:

$$\eta_{th} = 1 - \frac{1}{\sqrt{\theta_t}}$$

For $\theta_t = 6.25$, we should get $\eta_{th} = 1 - \frac{1}{2.5} = 6$ (again, within truncation error).

This invariance of η_{th} occurs despite the fact that the turbine now extracts less power from the core flow. The extra is simply left as energy available for conversion to jet kinetic power, and the total remains the same.

2) Why is *F* only different (lower) by 2.6% when u_6 is increased from 462 m/s to 626 m/s and u_8 decreased from 462 m/s to 429 m/s? This is an example of a common occurrence in optimization: By definition the maximized quantity has *zero* differential change around the optimum, and for finite changes, it is still true in general that:

$$\frac{dy}{y_{max}} \ll \frac{dx}{x_{opt}}$$

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