16.410-13 Recitation 10 Problems

Problem 1: Simplex Method

Part A

Solve the following two linear programs using the simplex method.

LP 1

$$
\begin{aligned}
\text{maximize} \quad & 3x_1 + 4x_2 \\
\text{subject to} \quad & x_1 + x_2 \leq 4 \\
& 2x_1 + x_2 \leq 5 \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

Solution First, we will convert this linear program into standard form. This LP is already a maximization problem. To transform the constraints into equality form, let us introduce two slack variables, x_3 and x_4. Then, after writing $-3x_1+4x_2+z=0$ for the objective function, we obtain the following system of equations:

$$
\begin{aligned}
-3x_1 - 4x_2 + z = 0, \\
x_1 + x_2 + x_3 = 4, \\
2x_1 + x_2 + x_4 = 5.
\end{aligned}
$$

Thus, initial simplex tableau is

	x_1	x_2	x_3	x_4	b
z	-3	-4			1
r_1	1	1	1		4
r_2	2	1		1	5

Now pick the variable that has the smallest constant in the upper row. In this case, it is x_2, which has a constant -4. This column is our pivot column. To find the pivot row. Divide all the entries in the pivot column corresponding to the constraints, i.e., r_1 and r_2, by the values in the constant column. For r_1 we get $\frac{4}{1} = 4$, and for r_2 we get $\frac{1}{5} = 5$. Pick the smallest of the two, which is 4. That is, r_1 is our pivot column; our pivot element is 1, which lies both on the pivot row and the pivot column. Now, perform linear operations to convert the pivot column into a unit column. Thus you should perform the following row operations:

$$
\begin{aligned}
R_z = R_z - 4R_{r_1}, \\
R_{r_2} = R_{r_2} - R_{r_1},
\end{aligned}
$$

which results in the following tableau:

	x_1	x_2	x_3	x_4	b
z	1		4		16
r_1	1	1	1		4
r_2	1		-1	1	1

We quickly realize that all the entries of the top column are all zeros. Hence, we have found a solution. The variables that are not assigned zero are those that correspond to a unit column. In this case, x_2 and x_4, which take values $x_2 = 4$ and $x_4 = 1$. The other variables take value zero, i.e., $x_1 = 0$ and $x_3 = 0$. Finally, the objective function is read as $z = 16$.

LP 2

$$\begin{aligned} \text{minimize} \quad & -2x_1 + x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 6 \\ & 3x_1 + 2x_2 \leq 12 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Solution This problem is a minimization problem. We will need to turn it into a maximization problem to represent it in the standard form. We multiple the objective function by -1 to obtain the maximization problem we are looking for.

$$\begin{aligned} \text{maximize} \quad & 2x_1 - x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 6 \\ & 3x_1 + 2x_2 \leq 12 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Introducing the slack variables x_3 and x_4 for the two constraints, our initial tableau is as follows:

	x_1	x_2	x_3	x_4	b
z	-2	1			0
r_1	1	2	1		6
r_2	3	2		1	12

We notice that the first column is the pivot column and r_2 is our pivot row. After appropriate algebraic manipulation to make the pivot column a unit column, we arrive at the following tableau:

	x_1	x_2	x_3	x_4	b
z		5/3		1/3	8
r_1		4/3	1	-1/3	2
r_2	1	2/3		1/3	4

We are done since the first row reads all positive values. Now, we read off the values as $x_1 = 4$ and $x_2 = 0$. The objective value for the maximization problem is $z = 8$. Hence, the objective value of the maximization problem is -8.

Part B

Consider the case when all the coefficients corresponding to the variables (and the slack variables) of the topmost row of your tableau equals to zero. What would that imply? Can you find an example?

Solution: That would imply the existence of infinitely many solutions (Why?).

For example, consider the following LP:

$$\begin{aligned} \text{maximize} \quad & x_1 + x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 1. \end{aligned}$$

The geometric representation of this LP is shown in the figure below. Clearly, any point along the bold line is a maximizes the objective function while respecting the constraints. Hence, this LP has infinitely many solutions. Let us show this by going through the steps of the simplex algorithm.

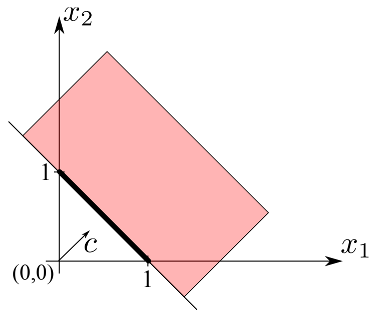

Figure 1: Geometric representation of the LP

We define a slack variable x_3 for the single constraint of the problem. The initial tableau is

	x_1	x_2	x_3	b
z	-1	-1	0	0
r_1	1	1	1	1

We pick the first column as the pivot column and the row labeled by r_1 as the pivot row. We do the algebraic operation required to make the pivot column a unit column, which results in the following tableau:

	x_1	x_2	x_3	b
z	0	0	0	1
r_1	1	1	1	1

Hence, we end up with the first row containing all zeros.

Problem 2: Transportation

Consider a network of mines/factories each of which use others products to produce their own. For instance, the iron ore produced by an iron ore mine is used to produce steel in a steel mill. This steel is used to produce mine wagons, which are then used by the iron ore mines. You would like to optimally coordinate the logistics operations between these mines/factories.

Assume that there are n factories, which are connected with roads. Nodes are assumed to be one way, since you are using certain transportation companies that only operate one way between the factories. A road from factory i to factory j is described by the pair (i, j). Denote the set of all roads by $\mathcal{A}$. Each (directed) road is operated by a single transportation company. The company operating road (i, j) charges $c_{i,j}$ dollars per each pound of cargo. Moreover, the company only has resources (e.g., trucks, trains) to carry up to $u_{i,j}$ tons of cargo per hour. Each factory k generates a certain product at a rate $b^{k,l}$ tons per hour that needs to reach factory l, either through a direct road (k, l) or through some other path in the road network. Products from the same origin can be split into parts and transported through different paths.

Formulate this problem as a linear programming problem. This problem is called the *multicommodity flow problem*, extensively studied in operations research. Similar problems are studied, e.g., for the optimization of future space logistics.

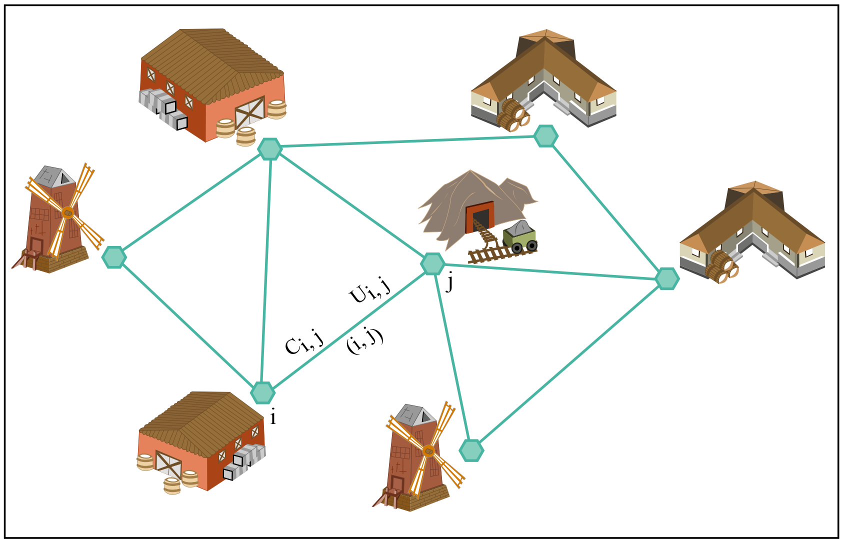

Image by MIT OpenCourseWare.

Figure 2: Multicommodity flow

Solution Introduce the variables $x_{i,j}^{k,l}$ that indicate the amount of product transported with origin k and destination l that traverses link (i,j). Then,

$$b_i^{l,k} = \begin{cases} b^{l,k}, & \text{if } i = k, \\ -b^{k,l}, & \text{if } i = l, \\ 0 & \text{otherwise.} \end{cases}$$

You can think of $b_i^{k,l}$ as the "net amount" of products going into factory site i coming from factory k and trying to reach factory k. Then, the following formulation can be used to solve the problem

$$\begin{array}{lrcll} \text{minimize} & \displaystyle\sum_{(i,j)\in\mathcal{A}} \sum_{k=1}^{n} \sum_{l=1}^{n} c_{i,j}\, x_{i,j}^{k,l} & & & \\ \text{subject to} & \displaystyle\sum_{\{j\,|\,(i,j)\in\mathcal{A}\}} x_{i,j}^{k,l} - \sum_{\{i\,|\,(i,j)\in\mathcal{A}\}} x_{i,j}^{k,l} & = & b_i^{k,l}, & \text{for all } i,k,l = 1,2,\ldots,n, \\ & \displaystyle\sum_{k=1}^{n} \sum_{l=1}^{n} x_{i,j}^{k,l} & \leq & u_{i,j}, & \text{for all } (i,j) \in \mathcal{A}, \\ & & & x_{i,j}^{k,l} \geq 0, & \text{for all } (i,j) \in \mathcal{A} \quad k,l = 1,2,\ldots,n. \end{array}$$

The first constraint is called the flow conservation constraint, which is very similar to the constraint we have introduced in the lecture when formulating the shortest path problem as a linear program. This flow conservation constraint ensures that any product coming into factory i goes out of the factory. That is, factory i does not "consume" a product unless the product was sent to factory i. The second constraint ensures operation within the constraints of cargo company operating a particular road.