16.346 Astrodynamics
Fall 2008

**Recall the Definitions**

Deviation in quantity measured: $\delta q$

Measurement vector: $\mathbf{b} = \begin{bmatrix} \mathbf{h} \\ \mathbf{0} \end{bmatrix}$

State vector deviation: $\delta\mathbf{x}(t) = \begin{bmatrix} \delta\mathbf{r}(t) \\ \delta\mathbf{v}(t) \end{bmatrix}$

Fundamental relationship: $\delta q = \mathbf{b}^{\mathsf{T}}\,\delta\mathbf{x}$

State transition matrix: $\boldsymbol{\Phi}(t_n, t_{n-1}) = \boldsymbol{\Phi}_{n,n-1}$

State vector deviations at $t_n$ and $t_{n-1}$: $\delta\mathbf{x}_n$ and $\delta\mathbf{x}_{n-1}$

Fundamental relationship: $\delta\mathbf{x}_n = \boldsymbol{\Phi}_{n,n-1}\,\delta\mathbf{x}_{n-1}$

**Effect at $t_n$ of observation made at $t_{n-1}$**

$$\delta q(t_{n-1}) = \delta q_{n-1} = \mathbf{b}_{n-1}^{\mathsf{T}}\,\delta\mathbf{x}_{n-1} = \mathbf{b}_{n-1}^{\mathsf{T}}\boldsymbol{\Phi}_{n,n-1}^{-1}\,\delta\mathbf{x}_n$$

**Recursive Formulation of the Navigation Algorithm**

$$\delta\widehat{\mathbf{x}}_n^* = \delta\widehat{\mathbf{x}}_n + \mathbf{w}(\delta\widetilde{q} - \delta\widehat{q}\,) \quad \text{where} \quad \delta\widehat{q} = \mathbf{b}^{\mathsf{T}}\,\delta\widehat{\mathbf{x}}_n \quad \text{and} \quad \delta\widehat{\mathbf{x}}_n = \boldsymbol{\Phi}_{n,n-1}\,\delta\widehat{\mathbf{x}}_{n-1}$$

## Propagating the Covariance Matrix $\mathbf{P}$ and the Error Transition Matrix $\mathbf{W}$

- **Using the state transition matrix**

$$\begin{aligned} \delta\widehat{\mathbf{x}}_{n-1} &= \delta\mathbf{x}_{n-1} + \mathbf{e}_{n-1} \\ \delta\widehat{\mathbf{x}}_n &= \boldsymbol{\Phi}_{n,n-1}\,\delta\widehat{\mathbf{x}}_{n-1} \\ \delta\mathbf{x}_n &= \boldsymbol{\Phi}_{n,n-1}\,\delta\mathbf{x}_{n-1} \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} \mathbf{e}_n &= \boldsymbol{\Phi}_{n,n-1}\mathbf{e}_{n-1} \\ \mathbf{e}_n^{\mathsf{T}} &= \mathbf{e}_{n-1}^{\mathsf{T}}\boldsymbol{\Phi}_{n,n-1}^{\mathsf{T}} \\ \overline{\mathbf{e}_n\mathbf{e}_n^{\mathsf{T}}} &= \boldsymbol{\Phi}_{n,n-1}\overline{\mathbf{e}_{n-1}\mathbf{e}_{n-1}^{\mathsf{T}}}\boldsymbol{\Phi}_{n,n-1}^{\mathsf{T}} \end{aligned}$$

Hence

$$\mathbf{P}_n = \boldsymbol{\Phi}_{n,n-1}\mathbf{P}_{n-1}\boldsymbol{\Phi}_{n,n-1}^{\mathsf{T}} \quad \text{and} \quad \mathbf{W}_n = \boldsymbol{\Phi}_{n,n-1}\mathbf{W}_{n-1}$$

- **Using differential equations**

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}\mathbf{x} \quad \Longrightarrow \quad \frac{d\mathbf{e}}{dt} = \mathbf{F}\mathbf{e} \quad \text{and} \quad \frac{d\mathbf{e}^{\mathsf{T}}}{dt} = \mathbf{e}^{\mathsf{T}}\mathbf{F}^{\mathsf{T}}$$

Hence

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}\mathbf{P} + \mathbf{P}\mathbf{F}^{\mathsf{T}} \quad \text{and} \quad \boxed{\frac{d\mathbf{W}}{dt} = \mathbf{F}\mathbf{W}}$$

## Encke's Method of Orbital Integration #9.4

### Deviations from the Osculating Orbit

Define

$$\mathbf{r}(t_0) = \mathbf{r}_{osc}(t_0) \qquad \mathbf{v}(t_0) = \mathbf{v}_{osc}(t_0)$$
$$\mathbf{r}(t) = \mathbf{r}_{osc}(t) + \boldsymbol{\delta}(t) \qquad \mathbf{v}(t) = \mathbf{v}_{osc}(t) + \boldsymbol{\nu}(t)$$

Then since

$$\frac{d^2\mathbf{r}}{dt^2} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{a}_d \qquad \frac{d^2\mathbf{r}_{osc}}{dt^2} + \frac{\mu}{r_{osc}^3}\mathbf{r}_{osc} = \mathbf{0}$$

we can write

$$\frac{d^2\boldsymbol{\delta}}{dt^2} + \frac{\mu}{r_{osc}^3}\boldsymbol{\delta} = \frac{\mu}{r_{osc}^3}\left(1 - \frac{r_{osc}^3}{r^3}\right)\mathbf{r} + \mathbf{a}_d$$

with the initial conditions

$$\boldsymbol{\delta}(t_0) = \mathbf{0} \qquad \text{and} \qquad \left.\frac{d\boldsymbol{\delta}}{dt}\right|_{t=t_0} = \boldsymbol{\nu}(t_0) = \mathbf{0}$$

### Coping with Numerical Accuracy for Small Deviations

When $r \approx r_{osc}$, we can define

$$q = \frac{(\boldsymbol{\delta} + 2\mathbf{r}_{osc}) \cdot \boldsymbol{\delta}}{r_{osc}^2}$$

and then write

$$\boxed{f(q) \stackrel{\text{def}}{=} 1 - \frac{r_{osc}^3}{r^3}} = 1 - (1+q)^{-\frac{3}{2}}$$
$$= 3 \cdot \frac{q}{2}\left[1 - \frac{5}{2}\left(\frac{q}{2}\right) + \frac{5 \cdot 7}{2 \cdot 3}\left(\frac{q}{2}\right)^2 - \frac{5 \cdot 7 \cdot 9}{2 \cdot 3 \cdot 4}\left(\frac{q}{2}\right)^3 + \cdots\right]$$

which is used in the classical method, or,

$$\boxed{f(q) = q\frac{3 + 3q + q^2}{(1+q)^{\frac{3}{2}} + (1+q)^3}}$$

as discovered by James E. Potter.

### Encke's Method — Johann Franz Encke (1791–1865)

**1.** Use the Lagrangian coefficients to extrapolate along the osculating orbit:

$$\boxed{\begin{aligned}\mathbf{r}_{osc}(t) &= F\mathbf{r}(t_0) + G\mathbf{v}(t_0)\\ \mathbf{v}_{osc}(t) &= F_t\mathbf{r}(t_0) + G_t\mathbf{v}(t_0)\end{aligned}}$$

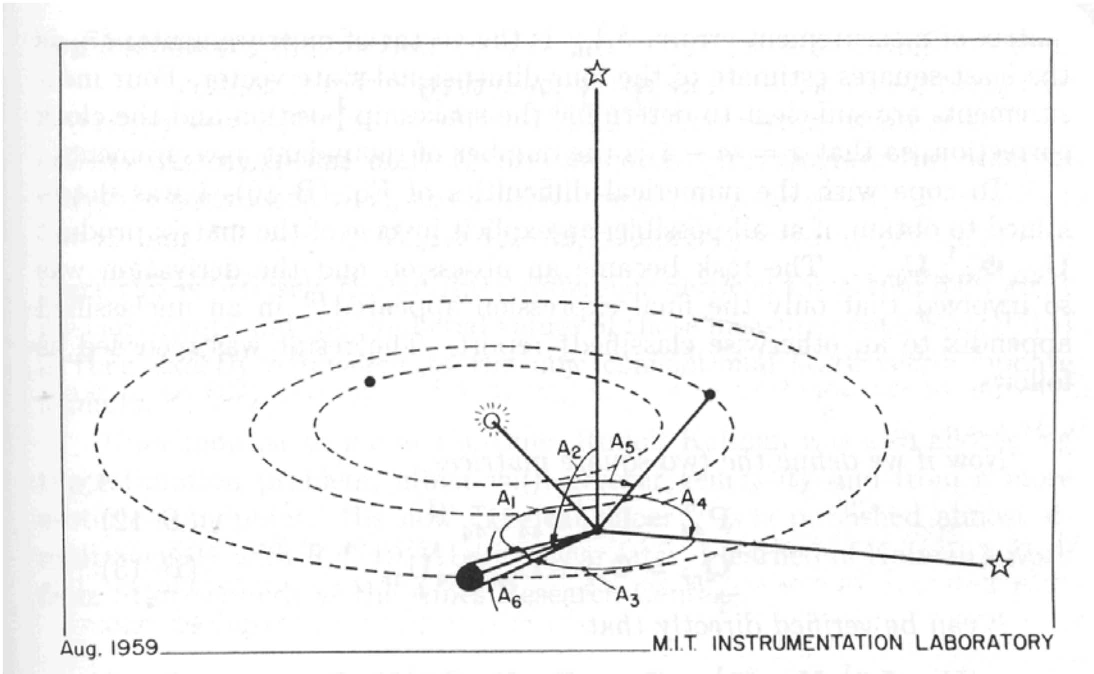
**Note:** Solving Kepler's equation is necessary to determine the coefficients.

**2.** Use numerical integration to propagate the deviation vector $\boldsymbol{\delta}$:

$$\boxed{\frac{d^2\boldsymbol{\delta}}{dt^2} + \frac{\mu}{r_{osc}^3}\boldsymbol{\delta} = \frac{\mu}{r_{osc}^3}f(q)\mathbf{r}(t) + \mathbf{a}_d} \qquad \text{where} \qquad \boxed{\mathbf{r} = \mathbf{r}_{osc} + \boldsymbol{\delta}}$$

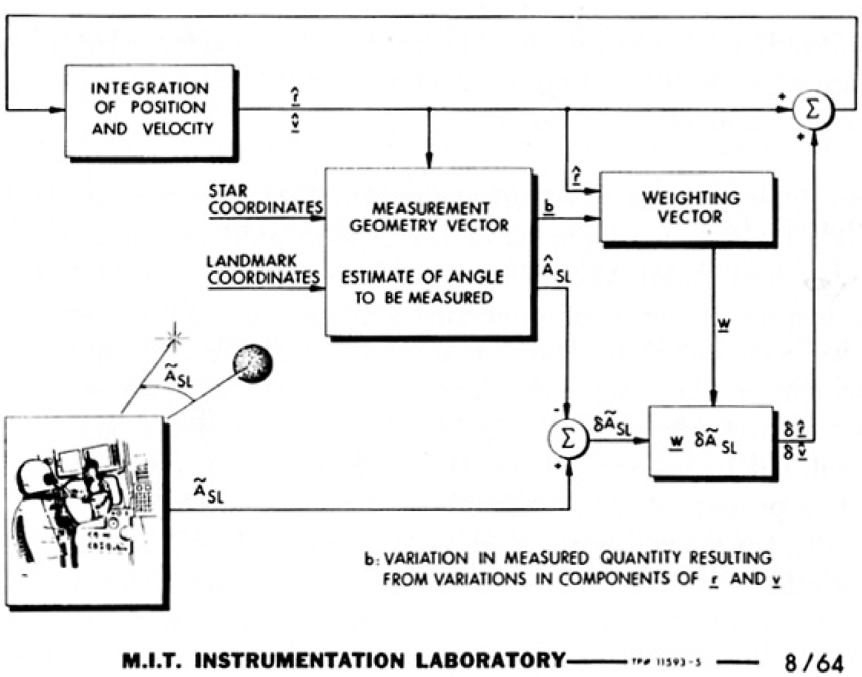
**3.** Use periodic *rectification* to maintain the efficiency of the algorithm.

## Navigating To Mars




Introduction Figure 7 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with

## Navigating to the Moon

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