16.346 Astrodynamics
Fall 2008

Orbital Parameter from the Semimajor Axis

Pages 274–275

If $\mathbf{r}_1 \times \mathbf{r}_2 \neq \mathbf{0}$, write $\mathbf{e} = A\,\mathbf{i}_{r_1} + B\,\mathbf{i}_{r_2}$

Then, from the equation of orbit $r = p - \mathbf{e}\cdot\mathbf{r}$ obtain

$$\begin{aligned} \mathbf{e}\cdot\mathbf{i}_{r_1} &= \frac{p}{r_1} - 1 = A + B\cos\theta \\ \mathbf{e}\cdot\mathbf{i}_{r_2} &= \frac{p}{r_2} - 1 = A\cos\theta + B \end{aligned} \quad\Longrightarrow\quad \begin{aligned} A\sin^2\theta &= \left(\frac{p}{r_1}-1\right) - \left(\frac{p}{r_2}-1\right)\cos\theta \\ B\sin^2\theta &= \left(\frac{p}{r_2}-1\right) - \left(\frac{p}{r_1}-1\right)\cos\theta \end{aligned}$$

Next
$$\mathbf{e}\cdot\mathbf{e} = e^2 = 1 - \frac{p}{a} = A^2 + 2AB\cos\theta + B^2$$

so that
$$\left(\frac{p}{r_1}-1\right)^2 - 2\left(\frac{p}{r_1}-1\right)\left(\frac{p}{r_2}-1\right)\cos\theta + \left(\frac{p}{r_2}-1\right)^2 = \left(1-\frac{p}{a}\right)\sin^2\theta$$

or, after simplification (using the trig formulas for the semiperimeter on Page 4),

$$\left(\frac{p}{p_m}\right)^2 - 2D\frac{p}{p_m} + 1 = 0 \qquad \text{where} \qquad D = \frac{r_1+r_2}{c} - \frac{r_1 r_2}{ac}\cos^2\tfrac{1}{2}\theta \equiv \frac{r_1+r_2}{c} - \frac{s(s-c)}{ac}$$

Therefore:
$$\boxed{\frac{p}{p_m} = D \pm \sqrt{D^2-1}}$$

Semimajor Axis from the Parameter

Alternately, from the quadratic equation, $1/a$ can be determined from p/p_m using

$$\frac{r_1+r_2}{c} - \frac{s(s-c)}{ac} = \boxed{D = \frac{1}{2}\left(\frac{p_m}{p} + \frac{p}{p_m}\right)} \quad\Longrightarrow\quad \boxed{\frac{1}{a} = \frac{r_1+r_2-cD}{s(s-c)}}$$

Semimajor Axis of the Minimum-Energy Orbit

Since the parameter of the minimum-energy orbit is $p = p_m$, then a_m can be determined from the last boxed equation (with $p/p_m = 1$). This equation becomes

$$D = 1 \qquad \text{or} \qquad a_m(r_1+r_2) - r_1 r_2\cos^2\tfrac{1}{2}\theta = a_m c$$

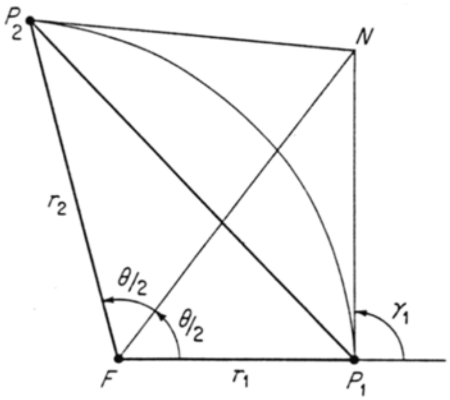
Introduce the semiperimeter of the triangle $s = \frac{1}{2}(r_1+r_2+c)$ and use one of the equations from Problem G–4 in Appendix G of the textbook to write

$$a_m(r_1+r_2-c) = \underbrace{r_1 r_2\cos^2\tfrac{1}{2}\theta = s(s-c)}_{\text{Problem G-4}} \qquad \text{or} \qquad 2a_m(s-c) = s(s-c)$$

Hence
$$\boxed{a_m = \tfrac{1}{2}s = \tfrac{1}{4}(r_1+r_2+c)}$$

Orbit Tangents and the Transfer-Angle Bisector #6.2

The line connecting the focus and the point of intersection of the orbital tangents at the terminals bisects the transfer angle.



$$\sqrt{r_1 r_2} = \begin{cases} FN \cos \frac{1}{2}(E_2 - E_1) & \text{ellipse} \\ FN & \text{parabola} \\ FN \cosh \frac{1}{2}(H_2 - H_1) & \text{hyperbola} \end{cases}$$

Fig. 6.7 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

Locus of the Eccentricity Vectors of the Boundary-Value Problem #6.3

Equation of orbit $\mathbf{e} \cdot \mathbf{r} = p - r$ at P_1 and P_2:

$$\begin{aligned} \mathbf{e} \cdot \mathbf{r}_1 &= p - r_1 \\ \mathbf{e} \cdot \mathbf{r}_2 &= p - r_2 \end{aligned} \quad \Longrightarrow \quad \mathbf{e} \cdot (\mathbf{r}_2 - \mathbf{r}_1) = r_1 - r_2 \quad \text{or} \quad \boxed{-\mathbf{e} \cdot \mathbf{i}_c = \frac{r_2 - r_1}{c}}$$

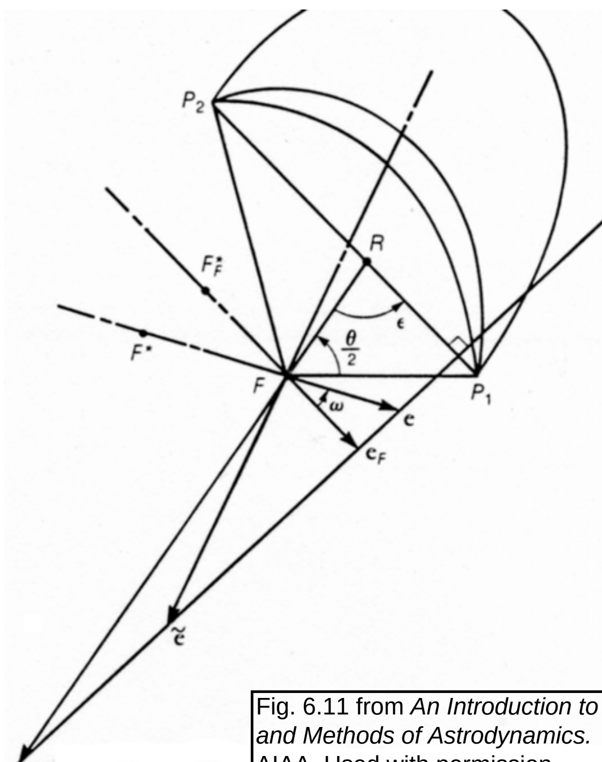


Fig. 6.11 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.

Locus of the Vacant Focus

Elliptic Orbits:
$$r_1 + P_1F^* = 2a \qquad r_2 + P_2F^* = 2a \quad \Longrightarrow \quad \boxed{P_2F^* - P_1F^* = -(r_2 - r_1) = 2a^*}$$

Note: $2a \geq r_1 + r_2 + c = 2s = 2a_m$

Hyperbolic Orbits:
$$r_1 - P_1F^* = 2a \qquad r_2 - P_2F^* = 2a \quad \Longrightarrow \quad \boxed{P_1F^* - P_2F^* = -(r_2 - r_1) = 2a^*}$$

Note: F_0^*: Rectilinear orbit from P_1 to P_2 with $a = 0$ and $e = \infty$

Note: $\widetilde{F}_0^*$: Two straight-line segments P_2 to F to P_1 with $a = 0$ and $e = \sec \frac{1}{2}\theta$

Hyperbolic locus of vacant foci:
$$\boxed{2a^* = -(r_2 - r_1) \qquad e^* = \frac{c}{r_2 - r_1}}$$

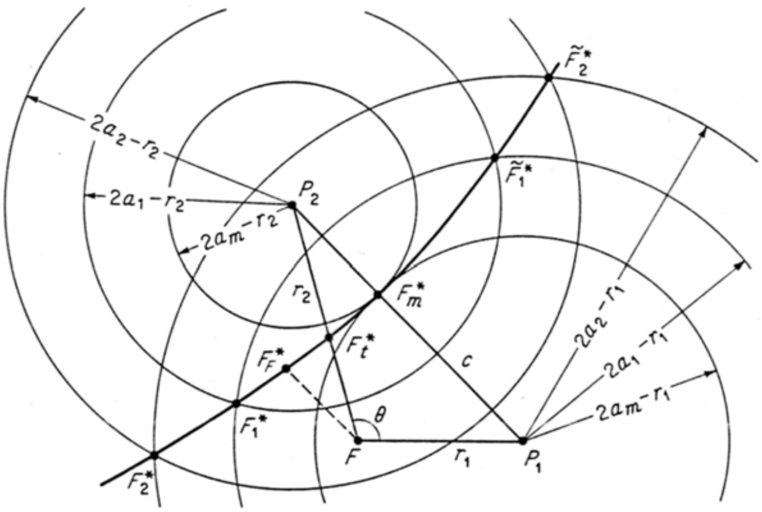
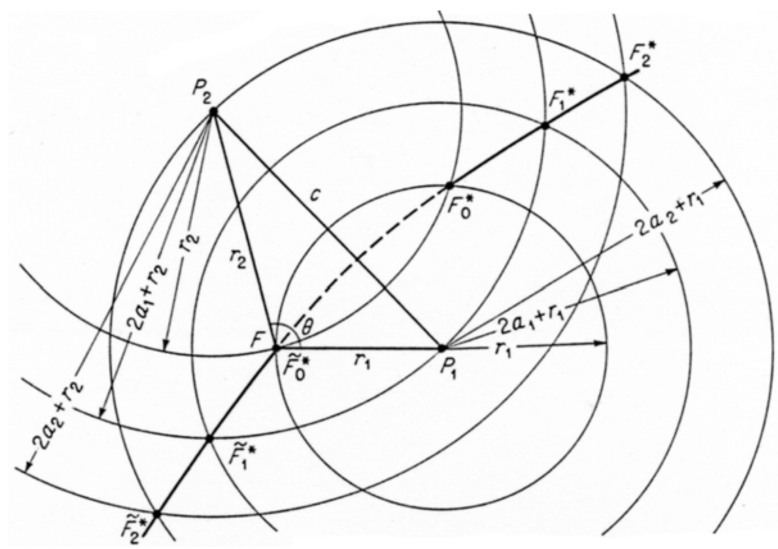


Fig. 6.17 and 6.18 from *An Introduction to the Mathematics and Methods of Astrodynamics.* Courtesy of AIAA. Used with permission.



The Semiperimeter of a Triangle

One of the twelve greatest theorems of all times, according to the author William Dunham in his book *Journey through Genius* published by John Wiley & Sons, Inc., was the formula for the area of a triangle involving only the lengths of the three sides which was discovered by either Archimedes (287 B.C. – 212 B.C.) or a centuary later, by Heron.

$$\boxed{\text{Area of a Triangle} = \sqrt{s(s-a)(s-b)(s-c)}}$$

where

$$\boxed{s = \tfrac{1}{2}(a+b+c)}$$

There are other remarkable formulas given in Appendix G on Page 364 of our textbook, namely, Problems G–6 and G–7.

Several trigonometric formulas given in Problem G-5 are useful for our work with the Boundary-Value Problem. In particular

$$\sin \tfrac{1}{2}\alpha = \sqrt{\frac{(s-b)(s-c)}{bc}} \qquad \text{and} \qquad \cos \tfrac{1}{2}\alpha = \sqrt{\frac{s(s-a)}{bc}}$$

Indeed, you might find it instructive to derive these and the equation

$$\sin\alpha = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}$$

The angle α is the angle opposite side a and between the sides b and c.