16.323 Principles of Optimal Control
Spring 2008

16.323 Handout #3
Prof. J. P. How March 6, 2008
Due: March 18, 2008

16.323 Homework Assignment #3

1. Find the curve $x^\star(t)$ that minimizes the functional

$$J(x) = \int_0^1 [\frac{1}{2}\dot{x}^2(t) + 5x(t)\dot{x}(t) + x^2(t) + 5x(t)]dt$$

and passes through the points $x(0) = 1$ and $x(1) = 3$

2. One important calculus of variations problem that we did not discuss in class has the same basic form, but with constraints that are given by an integral - called *isoperimetric constraints*:

$$\min J = \int_{t_0}^{t_f} g[\mathbf{x}, \dot{\mathbf{x}}, t]\ dt$$
$$\text{such that} : \int_{t_0}^{t_f} e[\mathbf{x}, \dot{\mathbf{x}}, t]\ dt = C$$

where we will assume that t_f is free but $\mathbf{x}(t_f)$ is fixed.

(a) Use the same approach followed in the notes to find the **necessary and boundary conditions** for this optimal control problem. In particular, augment the constraint to the cost using a constant Lagrange multiplier vector ν, and show that these conditions can be written in the form:

$$\begin{aligned} \frac{\partial g_a}{\partial \mathbf{x}} - \frac{d}{dt}\left(\frac{\partial g_a}{\partial \dot{\mathbf{x}}}\right) &= 0 \\ g_a(t_f) - \frac{\partial g_a}{\partial \dot{\mathbf{x}}}(t_f)\dot{\mathbf{x}}(t_f) &= 0 \\ \int_{t_0}^{t_f} e[\mathbf{x}, \dot{\mathbf{x}}, t]\ dt &= C \end{aligned}$$

where $g_a = g + \nu^T e$

(b) Use the results of part (a) to clearly state the differential equations and corresponding boundary conditions that must be solved to find the curve $y(x)$ of a specified length L with endpoints on the x-axis (i.e., at $x = 0$ and $x = x_f$) that encloses the maximum area, so that

$$J = \int_0^{x_f} y dx \qquad \text{and} \qquad \int_0^{x_f} \sqrt{1 + \dot{y}^2}\ dx = L$$

with x_f free.

3. Consider the unstable second order system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_2 + u\end{aligned}$$

and performance index

$$J = \int_0^\infty (R_{xx} x_1^2 + R_{uu} u^2)\, dt$$

   (a) For $R_{xx}/R_{uu} = 1$ show analytically (i.e. not using Matlab) that the steady-state LQR gains are:

   $$K = [\ 1 \quad \sqrt{3}+1\ ]$$

   and that the closed-loop poles are at $s = -(\sqrt{3} \pm j)/2$.

   (b) Using the steady-state regulator gains from Matlab in each case, plot (on one graph) the closed-loop locations for a range of possible values of R_{xx}/R_{uu}. Show the pole locations for the expensive control problem. Do you see any trends here?

4. Find the Hamiltonian and then solve the necessary conditions to compute the optimal control and state trajectory that minimize

$$J = \int_0^1 u^2 dt$$

for the system $\dot{x} = -2x + u$ with initial state $x(0) = 2$ and terminal state $x(1) = 0$. Plot the optimal control and state response using Matlab.

5. Consider a disturbance rejection problem that minimizes:

$$J = \frac{1}{2}\mathbf{x}(t_f)^T H \mathbf{x}(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \mathbf{x}^T(t) R_{xx}(t)\mathbf{x}(t) + \mathbf{u}(t)^T R_{uu}(t)\mathbf{u}(t)\ dt \tag{1}$$

subject to

$$\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t) + \mathbf{w}(t). \tag{2}$$

To handle the disturbance term, the optimal control should consist of both a feedback term and a feedforward term (assume $\mathbf{w}(t)$ is known).

$$\mathbf{u}^\star(t) = -K(t)\mathbf{x}(t) + \mathbf{u}_{fw}(t), \tag{3}$$

Using the Hamilton-Jacobi-Bellman equation, show that a possible optimal value function is of the form

$$J^\star(\mathbf{x}(t), t) = \frac{1}{2}\mathbf{x}^T(t)P(t)\mathbf{x}(t) + b^T(t)\mathbf{x}(t) + \frac{1}{2}c(t), \tag{4}$$

where

$$K(t) = R_{uu}^{-1}(t)B^T(t)P(t), \quad \mathbf{u}_{fw} = -R_{uu}^{-1}(t)B^T(t)b(t) \tag{5}$$

In the process demonstrate that the conditions that must be satisfied are:

$$\begin{aligned}-\dot{P}(t) &= A^T(t)P(t) + P(t)A(t) + R_{xx}(t) - P(t)B(t)R_{uu}^{-1}(t)B^T(t)P(t) \\ \dot{b}(t) &= -\left[A(t) - B(t)R_{uu}^{-1}(t)B^T(t)P(t)\right]^T b(t) - P(t)\mathbf{w}(t) \\ \dot{c}(t) &= b^T(t)B(t)R_{uu}^{-1}(t)B^T(t)b(t) - 2b^T(t)\mathbf{w}(t).\end{aligned}$$

with boundary conditions: $P(t_f) = H$, $b(t_f) = 0$, $c(t_f) = 0$.