Lecture 14

Last time: $w(t,\tau) \Rightarrow w(t-\tau)$

$$y(t) = \int_{-\infty}^{t} w(t-\tau)x(\tau)d\tau$$

$$\text{Let:} \begin{cases} \tau' = t-\tau \\ -d\tau = d\tau' \end{cases}$$

$$y(t) = \int_{0}^{\infty} w(\tau')x(t-\tau')d\tau'$$

For the differential system characterized by its equations of state, specialization to invariance means that the system matrices A, B, C are constants.

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$
$$\underline{y} = C\underline{x}$$

For A, B, C constant:

$$\underline{y}(t) = C\underline{x}(t)$$

$$\underline{x}(t) = \Phi(t-t_0)\underline{x}(t_0) + \int_{t_0}^{t} \Phi(t-\tau)B\underline{u}(\tau)d\tau$$

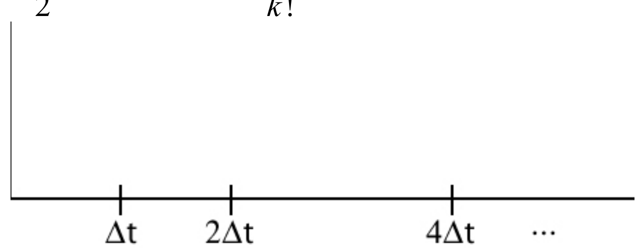
The transition matrix can be expressed analytically in this case.

$$\frac{d}{dt}\Phi(t,\tau) = A\Phi(t,\tau), \quad \text{where } \Phi(\tau,\tau) = I$$

This is a matrix form of first order, constant coefficient differential equation. The solution is the matrix exponential.

$$\Phi(t,\tau) = e^{A(t-\tau)}$$

$$e^{A(t-\tau)} = I + A(t-\tau) + \frac{1}{2}A^2(t-\tau)^2 + \ldots + \frac{1}{k!}A^k(t-\tau)^k + \ldots$$



Useful for computing $\Phi(t)$ for small enough $t-\tau$.

The solution is

$$\underline{y}(t) = C\underline{x}(t)$$

$$\underline{x}(t) = e^{A(t-t_0)}\underline{x}(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}B\underline{u}(t)d\tau$$

For $t_0 \to \infty$:

$$\underline{x}(t) = \int_{-\infty}^{t} e^{A(t-\tau)}B\underline{u}(\tau)d\tau$$

$$= \int_{0}^{\infty} e^{A\tau'}B\underline{u}(t-\tau')d\tau'$$

and for a single input, single output (SISO) system,

$$w(t) = \underline{c}^T e^{At}\underline{b}$$

If $x(t) = e^{j\omega t}$ for all past time

$$y(t) = \int_{0}^{\infty} w(\tau)e^{j\omega(t-\tau)}d\tau$$

$$= \left[\int_{0}^{\infty} w(\tau)e^{-j\omega\tau}d\tau\right]e^{j\omega t}$$

$$= F(\omega)x(t)$$

Since $w(\tau) = 0$ for $\tau < 0$ for a realizable system, we see that the *steady state sinusoidal response function,* $F(\omega)$, for a system is the Fourier transform of the weighting function – where the coefficient unity must be used.

$$F(\omega) = \int_{-\infty}^{\infty} w(\tau)e^{-j\omega\tau}d\tau$$

and $w(\tau)$ for a stable system is Fourier transformable.
Then

$$w(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

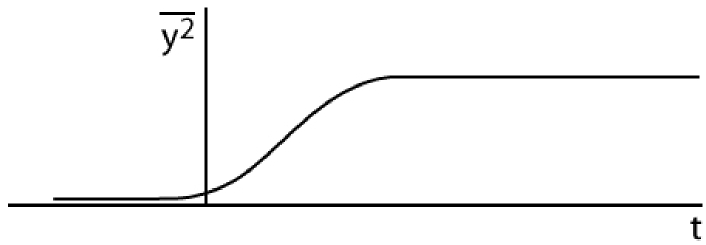
You can compute the response to any input at all, including transient responses, having defined $F(\omega)$ for all frequencies.

The *static sensitivity* of the system is the zero frequency gain, $F(0)$, which is just the integral of the weighting function.

$$F(0) = \int_{0}^{\infty} w(\tau)d\tau$$

Stationary statistics

Invariant output statistics implies more than stationary inputs and invariant systems; it also implies that the system has been in operation long enough under the present conditions to have exhausted all transients.



Input-Output Relations for Correlation and Spectral Density Functions

Derive autocorrelation of output in terms of autocorrelation of input

$$y(t) = \int_0^\infty w(\tau_1)x(t-\tau_1)d\tau_1$$

$$\bar{y} = \int_0^\infty w(\tau_1)\bar{x}d\tau_1$$

$$= \bar{x}\int_0^\infty w(\tau_1)d\tau_1$$

$$R_{yy}(\tau) = \overline{y(t)y(t+\tau)}$$

$$= \int_0^\infty d\tau_1 w(\tau_1)\int_0^\infty d\tau_2 w(\tau_2)\overline{x(t-\tau_1)x(t+\tau-\tau_2)}$$

$$= \int_0^\infty d\tau_1 w(\tau_1)\int_0^\infty d\tau_2 w(\tau_2)R_{xx}(\tau+\tau_1-\tau_2)$$

$$\overline{y^2} = \int_0^\infty d\tau_1 w(\tau_1)\int_0^\infty d\tau_2 w(\tau_2)R_{xx}(\tau_1-\tau_2)$$

$$R_{xy}(\tau) = \overline{x(t)y(t+\tau)}$$

$$= \overline{x(t)\int_0^\infty w(\tau_1)x(t+\tau-\tau_1)d\tau_1}$$

$$= \int_0^\infty w(\tau_1)R_{xx}(\tau-\tau_1)d\tau_1$$

<u>Transform to get power density spectrum of output.</u>

$$\overline{y} = \overline{x}\int_0^\infty w(\tau)d\tau$$
$$= F(0)\overline{x}$$

$$S_{yy}(\omega) = \int_{-\infty}^{\infty} R_{yy}(\tau)e^{-j\omega\tau}d\tau$$
$$= \int_{-\infty}^{\infty} d\tau \int_0^\infty d\tau_1 w(\tau_1) \int_0^\infty d\tau_2 w(\tau_2) R_{xx}(\tau + \tau_1 - \tau_2)e^{-j\omega\tau_1}$$
$$= \underbrace{\int_{-\infty}^{\infty} d\tau R_{xx}(\tau + \tau_1 - \tau_2)e^{-j\omega(\tau+\tau_1-\tau_2)}}_{\text{first integral}} \int_0^\infty d\tau_1 w(\tau_1)e^{j\omega\tau_1} \int_0^\infty d\tau_2 w(\tau_2)e^{-j\omega\tau_2}$$

In first integral only, let $\begin{cases} \tau' = \tau + \tau_1 - \tau_2 \\ d\tau' = d\tau \end{cases}$

$$S_{yy}(\omega) = \int_{-\infty}^{\infty} d\tau' R_{xx}(\tau')e^{-j\omega\tau'} \int_{-\infty}^{\infty} d\tau_1 w(\tau_1)e^{j\omega\tau_1} \int_{-\infty}^{\infty} d\tau_2 w(\tau_2)e^{-j\omega\tau_2}$$
$$= S_{xx}(\omega)F(-\omega)F(\omega)$$
$$= \left|F(\omega)\right|^2 S_{xx}(\omega)$$

The power spectral density thus does not depend upon phase properties.

The *cross-spectral density function* can be derived similarly, to obtain:
$S_{xy}(\omega) = F(\omega)S_{xx}(\omega)$

<u>Mean squared output in time and frequency domain</u>

$$\overline{y^2} = R_{yy}(0) = \int_0^\infty d\tau_1 w(\tau_1) \int_0^\infty d\tau_2 w(\tau_2) R_{xx}(\tau_1 - \tau_2)$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{yy}(\omega)d\omega$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)F(-\omega)S_{xx}(\omega)d\omega$$

Generally speaking, with linear invariant systems it is easier to work in the transform domain than the time domain – so we shall commonly use the last expression to calculate the mean squared output of a system. However, control engineers are more accustomed to working with Laplace transforms than with Fourier transforms. By making the change of variables $s = j\omega$ we can cast this expression in that form.

$$\overline{y^2} = \frac{1}{2\pi}\int_{-j\infty}^{j\infty} F\left(\frac{s}{j}\right)F\left(-\frac{s}{j}\right)S_{xx}\left(\frac{s}{j}\right)\frac{ds}{j}$$
$$= \frac{1}{2j\pi}\int_{-\infty}^{\infty} F'(s)F'(-s)S'_{xx}(s)ds$$

We know that $S_{xx}(\omega)$ is even. If it is a rational function of ω, and we will work exclusively with rational spectra, it is then a rational function of ω^2. So only even powers of ω appear in $S_{xx}(\omega)$ and thus $S_{xx}\left(\frac{s}{j}\right)$ which we may call $S_{xx}(s)$ is derived from $S_{xx}(\omega)$ by replacing ω^2 by $-s^2$.

$F'(s)$ is the ordinary transfer function of the system – the Laplace transform of its weighting function. Because $w(t)=0,\ t<0$.

We shall drop the primes from now on.

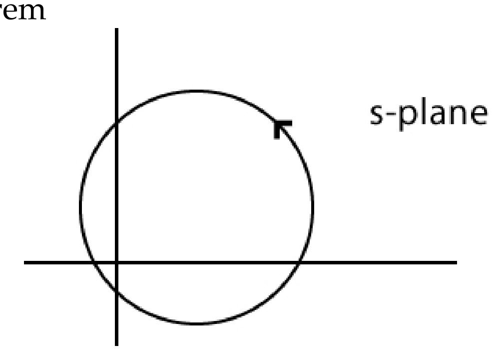
$$\overline{y^2} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} F(s)F(-s)S_{xx}(s)ds$$

$$\left.\begin{aligned}\omega^2 &= -s^2\\ \omega^4 &= s^4\end{aligned}\right\}\text{ in } S_{xx}(s)$$

Integrating the output spectrum

General method

Cauchy Residue Theorem



$$\oint_C F(s)ds = 2\pi j\sum\left(\text{residues of } F(s) \text{ at the poles enclosed in the contour C}\right)$$

If $F(s)$ has a pole of order m at $z=a$,

$$\text{Res}(a) = \frac{1}{(m-1)!}\left\{\frac{d^{m-1}}{ds^{m-1}}\left[(s-a)^m F(s)\right]_{s=a}\right\}$$

$F(s)$ has a pole of order m at $s=a$ if m is the smallest integer for which

$$\lim_{s \to a}\left[(s-a)^m F(s)\right]$$

is finite.

If $F(s)$ is rational and has a 1st order pole at a,

$$F(s) = \frac{N(s)}{D(s)}$$

$$= \frac{N(s)}{(s-a)(s-b)...}$$

then

$$\text{Res}(a)_{m=1} = \lim_{s \to a}\left[(s-a)F(s)\right]$$

$$= \frac{N(a)}{(a-b)(a-c)...}$$