

16.225 - Computational Mechanics of Materials
Homework assignment # 2
Handed out: 9/25/03
Due: 10/8/03

The main objective of this assignment is to add capability to your version of `sumMIT` to solve linear elasticity problems. Details of the data structures and additional aspects of the program will be given in recitation.

1. Write a 3-node simplex element `elmt01` to do plane stress/strain linear elasticity. The material constant array may be taken as: $d(1, 1) =$ Young's modulus; $d(2, 1) =$ Poisson's ratio; $d(3, 1) = 0 \Rightarrow$ plane strain, $d(3, 1) = 1 \Rightarrow$ plane stress.
2. Verify that your the element and, thus, your implementation, satisfies the patch test. To this end, consider the unit square $[0, 1]^2$ subjected to uniform tractions: a) σ_{11} ; b) σ_{22} , and c) σ_{12} . Mesh the square by means of one node at each of its corners and one additional interior node. This gives a mesh of 4 elements and 5 nodes. The interior node should be off-center at an arbitrary position so that the elements are randomly distorted. Verify that the displacements returned by your code are exact.
3. iv) Consider an isotropic linear elastic square plate of width $2L$. The center of the plate is at the origin and its sides are aligned with the coordinate axes. The plate contains a small circular hole of radius $R \ll L$ at its center, and is subjected to a remote uniform stress $\sigma_{11} = \sigma^\infty$ on its boundary. For an infinite plate, $L \rightarrow \infty$, the exact solution of this problem is (e. g., I. S. Sokolnikoff, *Mathematical Theory of Elasticity*, McGraw-Hill, 1956, p. 291):

$$\begin{aligned}
u_r &= \frac{\sigma^\infty}{8\mu r} \left\{ (\kappa - 1)r^2 + 2R^2 + 2 \left[R^2(\kappa + 1) + r^2 - \frac{R^4}{r^2} \right] \cos 2\theta \right\}, \\
u_\theta &= -\frac{\sigma^\infty}{4\mu r} \left[r^2 + R^2(\kappa - 1) + \frac{R^4}{r^2} \right] \sin 2\theta, \\
\sigma_{rr} &= \frac{\sigma^\infty}{2} \left[\left(1 - \frac{R^2}{r^2} \right) + \left(1 - \frac{4R^2}{r^2} + \frac{3R^4}{r^4} \right) \cos 2\theta \right], \\
\sigma_{\theta\theta} &= \frac{\sigma^\infty}{2} \left[\left(1 + \frac{R^2}{r^2} \right) - \left(1 + \frac{3R^4}{r^4} \right) \cos 2\theta \right], \\
\sigma_{r\theta} &= -\frac{\sigma^\infty}{2} \left(1 + \frac{2R^2}{r^2} - \frac{3R^4}{r^4} \right) \sin 2\theta.
\end{aligned} \tag{1}$$

where $\kappa = 3 - 4\nu$, μ is the shear modulus, ν Poisson's ratio, r is the distance to the origin and the polar angle θ is measured from the x_1 -axis. Note that you only need consider one quadrant of the plate, e. g., $x_1 \geq 0$, $x_2 \geq 0$ due to the symmetry of the problem, provided that you impose the appropriate boundary conditions $u_1 = 0$ on edge $x_1 = 0$ and $u_2 = 0$ on edge $x_2 = 0$. Compute the finite element solution for meshes corresponding to values of the mesh scaling factor $h = 1/2^p$ with $p = 0, \dots, 5$. Plot the deformed mesh and the contours of σ_{11}, σ_{22} . Also plot the energy-norm error $\| \mathbf{u}_h - \mathbf{u} \|_E$, computed by numerical quadrature, as a function of the number of elements E in $\log - \log$ scale.