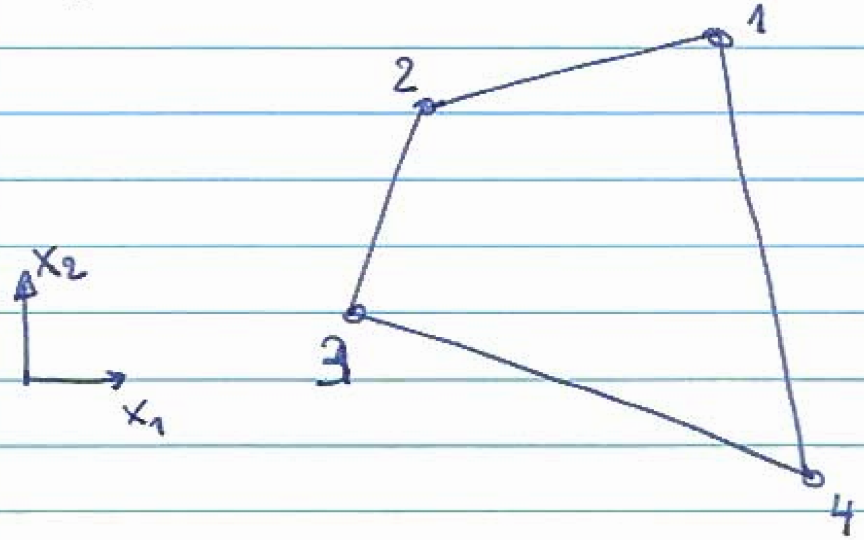
Formulation of isoparametric elements

(Bathe's book)

Consider the quadrilateral element shown in the figure:



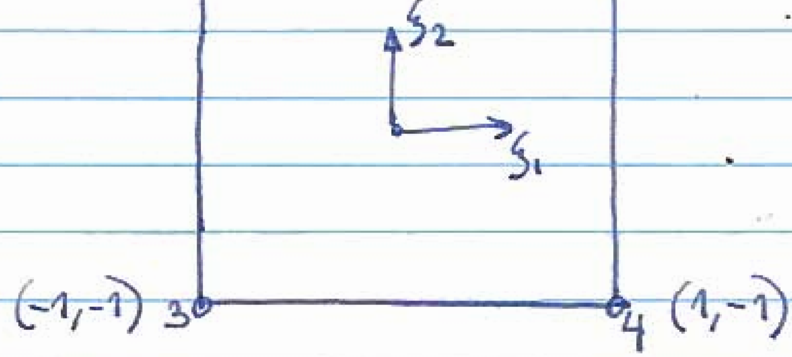
nodal coordinates:

X_i^a, $i=1,2$, $a=1,\ldots,4$

nodal displacements

U_i^a

We need to generalize our interpolation for the linear square. Consider the mapping from the following master element



We interpolate the displacement field as before

$$u = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [H]\{U\}$$, where U are the nodal displacements

$$H = \begin{bmatrix} \frac{1}{2}(1+\xi_1)(1+\xi_2) & 0 & \cdots\cdots \\ 0 & \frac{1}{2}(1+\xi_1)(1+\xi_2) & \cdots\cdots \end{bmatrix}$$

So $\boxed{u = u(\xi_1, \xi_2)}$

But we need expression for $\frac{\partial u_1}{\partial x_1}, \ldots$, (i.e. strains).

We follow the following procedure:

- Define the mapping (interpolation) from the master element to the quadrilateral:

$$\boxed{x(\xi) = \begin{Bmatrix} x_1(\xi_1, \xi_2) \\ x_2(\xi_1, \xi_2) \end{Bmatrix} = [H]\{X\}}$$

where $\{X\}$ is the vector of nodal coordinates

$$\{X\}^T = \{X_1^1 \; X_2^1 \; X_1^2 \; X_2^2 \cdots\cdots X_1^4 \; X_2^4\}$$

- Link derivatives through chain rule

$$\frac{\partial f}{\partial \xi_1} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial \xi_1} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial \xi_1}$$

$$\frac{\partial f}{\partial \xi_2} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial \xi_2} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial \xi_2}$$

In vector form:

$$\begin{Bmatrix} \frac{\partial f}{\partial \xi_1} \\ \frac{\partial f}{\partial \xi_2} \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_1} \\ \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_2} \end{bmatrix}}_{J} \begin{Bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{Bmatrix}$$

What we really need is the inverse:

$$\begin{Bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\partial f}{\partial \xi_1} \\ \frac{\partial f}{\partial \xi_2} \end{Bmatrix} , \quad J^{-1} = \frac{1}{\det J} \begin{bmatrix} \frac{\partial x_2}{\partial \xi_2} & -\frac{\partial x_2}{\partial \xi_1} \\ -\frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_1} \end{bmatrix}$$

- Now we can derive the interpolation of strains

$$\varepsilon = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{Bmatrix} \partial u_1/\partial x_1 \\ \partial u_2/\partial x_2 \\ \partial u_1/\partial x_2 + \partial u_2/\partial x_1 \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial u_1}{\partial x_1} \\ \dfrac{\partial u_1}{\partial x_2} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \dfrac{\partial u_1}{\partial \xi_1} \\ \dfrac{\partial u_1}{\partial \xi_2} \end{Bmatrix}$$, same for u_2

The strain vector can then be written as:

$$\underset{}{\{\varepsilon\}} = \underset{3\times 4}{[A]} \underset{4\times 1}{\begin{Bmatrix} \partial u_1/\partial \xi_1 \\ \partial u_1/\partial \xi_2 \\ \partial u_2/\partial \xi_1 \\ \partial u_2/\partial \xi_2 \end{Bmatrix}}$$, where

$$[A] \equiv \frac{1}{\det J} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix}$$ and

$$= \underset{4\times 8}{[G]} \underset{8\times 1}{\{U\}}$$, $[G]$ is the matrix of derivatives of our shape functions

$$\Rightarrow \overset{3\times1}{\{\varepsilon\}} = \underbrace{\overset{3\times4}{[A]}\,\overset{4\times8}{[G]}}_{\substack{B\\3\times8}}\,\overset{8\times1}{\{U\}}$$

- The final step is the computation of the stiffness matrix for the element:

$$[K^e] = \int_{\Omega^e} B^T C B \, dV$$

$$\boxed{[K^e] = \int_{-1}^{1}\int_{-1}^{1} B^T C B \, J \, d\xi_1 \, d\xi_2}$$

- Element force vector

$$\boxed{\{R^e\} = \int_{\Omega^e} [H]^T \{f\} \, dv + \int_{S^e} [H]^T \{t\} \, ds}$$