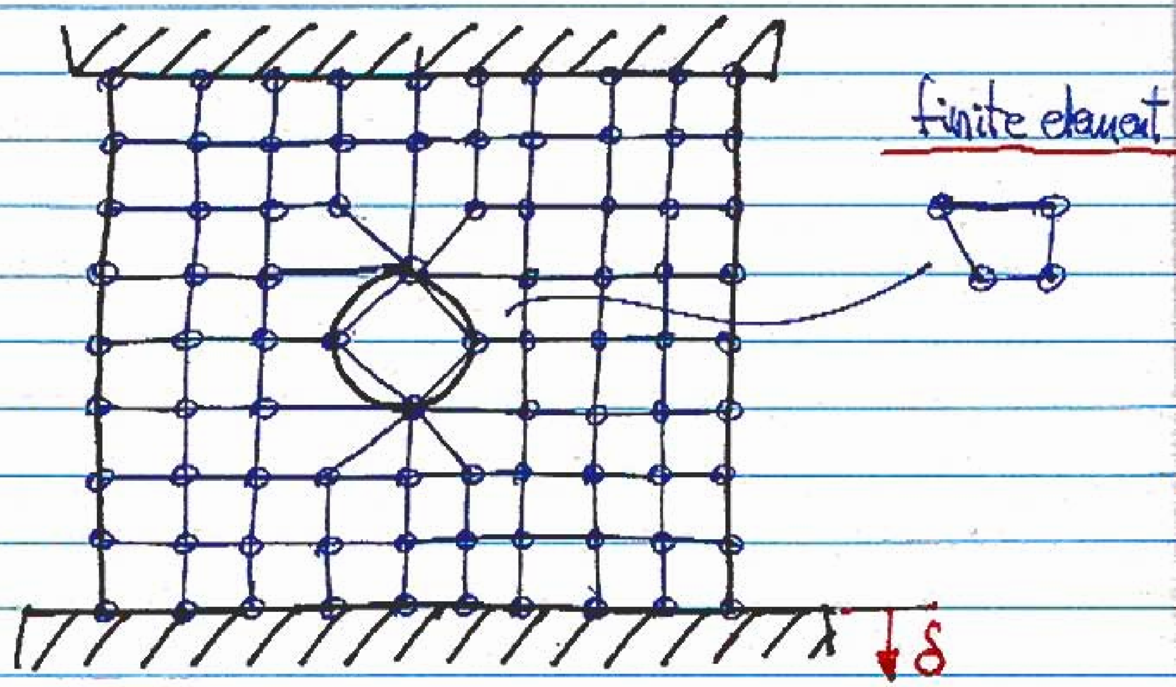
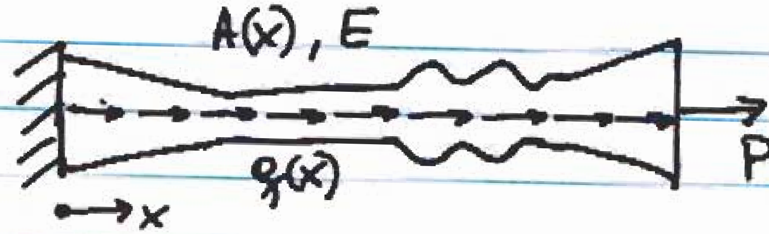
The Finite Element Method

- Overcome limitations of Ritz:
- Simple basis functions (low order polynomials)
- Basis functions supported in subdomains (finite elements)
- Basis functions constructed to provide interpolant of approximate solution.
- Undetermined parameters represent values of dependent variables (solution) at subdomain boundaries.

Mesh:



Model problem:



$$\frac{d}{dx}\left(EA(x)\,\frac{du}{dx} \right) + q(x) = 0 \qquad 0 < x < L$$

BC: "u" specified: $u(0) = 0$

"$EA\frac{du}{dx}$" specified: $\left. EA\frac{du}{dx} \right|_L = P$

Formulation of generic element:

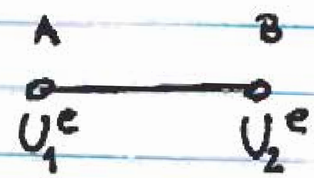
Ω^e: (x_A, x_B)
(x_1^e, x_2^e)

Seek variational approximation in this domain:

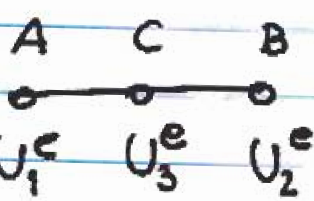
$$\boxed{u(x) \approx u_e(x) = \sum_{i=1}^{n} \phi_i^e(x)\, U_i^e \qquad x_A < x < x_B}$$

n: number of "nodes" in element

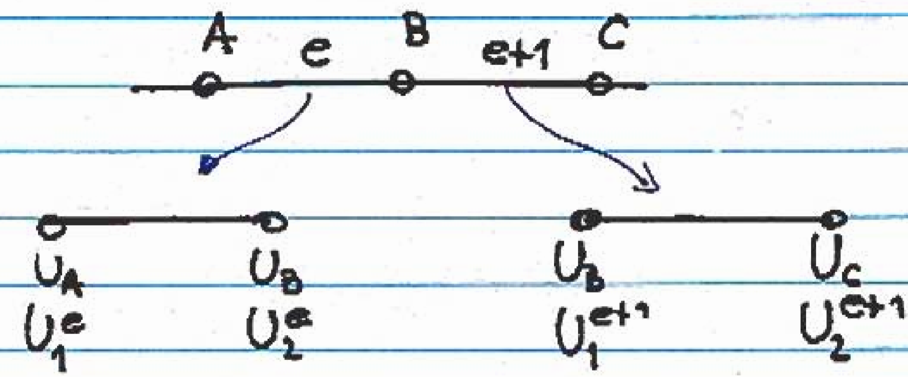
$n=2$:



$n=3$:



- **Note:** choosing undetermined parameters the values of the solution at the nodes enforces continuity of the solution across elements.



This imposes conditions on ϕ_i^e:

$$u_e(x_j) = \sum_{i=1}^{n} \phi_i^e(x_j)\, U_i^e = U_j^e$$

$$= \underbrace{\phi_1^e(x_j)}_{0} U_1^e + \underbrace{\dots}_{0} + \underbrace{\phi_j^e(x_j)}_{1} U_j^e + \underbrace{\dots}_{0} + \underbrace{\phi_n^e(x_j)}_{0} U_n^e$$

or • $\boxed{\phi_i^e(x_j) = \delta_{ij}}$

$\Rightarrow$ Lagrange polynomials

$$\boxed{\phi_j^e = \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{x - x_k^e}{x_j^e - x_k^e}}$$

$$\phi_j^e(x) = \frac{(x-x_1)}{(x_j-x_1)} \cdots \frac{(x-x_{j-1})}{(x_j-x_{j-1})} \Big[\qquad \Big] \frac{(x-x_{j+1})}{(x_j-x_{j+1})} \cdots \frac{(x-x_n)}{(x_j-x_n)}$$

$$\phi_j^e(x_i) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Another important property that the ϕ_i^e must satisfy is to allow for the representation of constant solutions exactly:

$$u(x) = C = \sum_{i=1}^{n} \phi_i^e(x)\, U_i^e \qquad x_A < x < x_B$$

$$= \sum_{i=1}^{n} \phi_i^e(x)\, C = C \underbrace{\sum_{i=1}^{n} \phi_i^e}_{1}$$

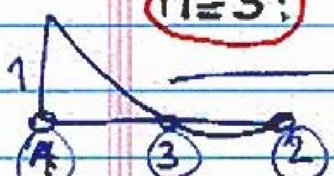
- $\boxed{\sum \phi_i^e(x) = 1}$

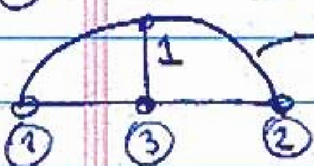
Examples:

n=2: $\phi_j^e(x) = \prod_{\substack{k=1 \\ k=j}}^{2} \frac{x-x_k}{x_j-x_k}$

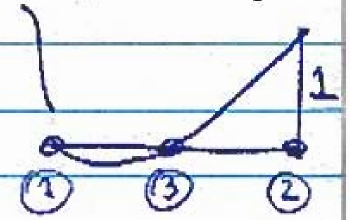
$$\phi_1^e = \frac{x - x_2}{x_1 - x_2} \quad ; \quad \phi_2^e = \frac{x - x_1}{x_2 - x_1}$$

n=3: $\phi_1^e = \dfrac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$ $\quad \phi_2^e = \dfrac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$



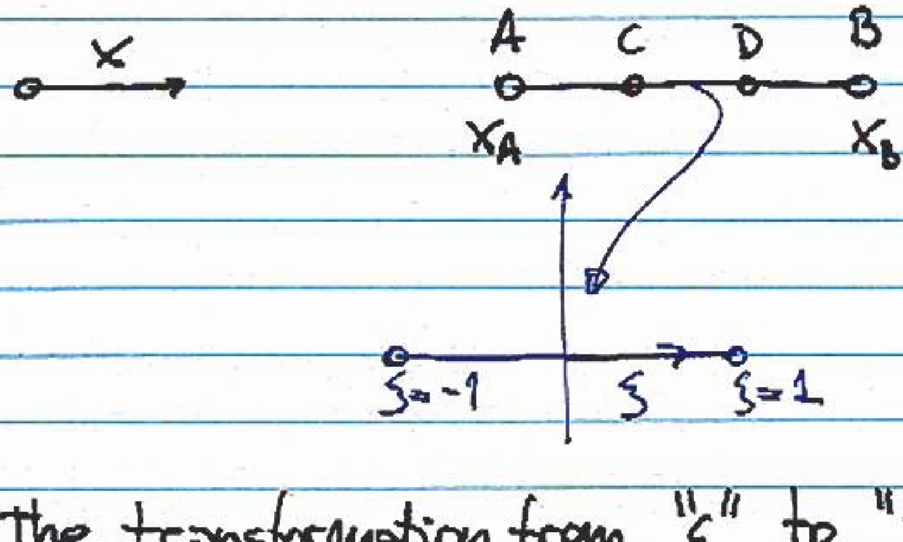


$$\phi_3^e = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$



Natural coordinate system

The functions look simpler when expressed in terms of the local coordinate system:



the transformation from "ξ" to "x" is

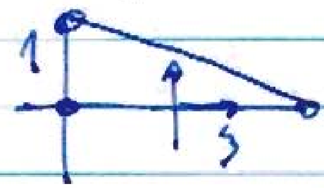
$$x = \frac{1-\xi}{2} x_A + \frac{1+\xi}{2} x_B$$

$n=2$:

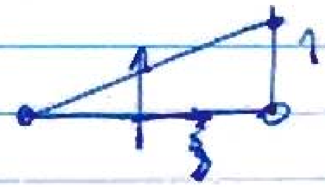
$$\phi_1 = \frac{x - x_B}{x_A - x_B} = \frac{1}{x_A - x_B}\left[\frac{x_A + x_B}{2} + \frac{x_B - x_A}{2}\xi - x_b\right]$$

$$= \frac{1}{x_A - x_B}\left[\frac{x_A - x_B}{2} - \frac{x_A - x_B}{2}\xi\right]$$

$$= \frac{1}{2}\left(1 - \xi\right)$$



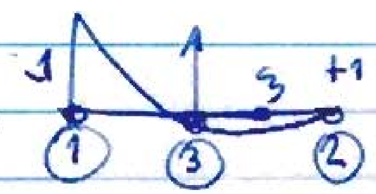
$$\phi_2 = \frac{1}{2}\left(1 + \xi\right)$$

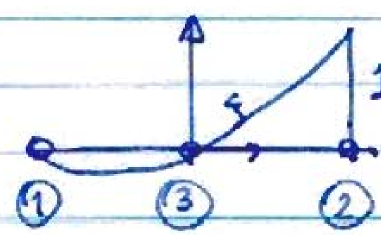


$n=3$:

$$\phi_2 = \frac{1}{2}\xi\left(\xi + 1\right)$$

$$\phi_1 = \frac{1}{2}\xi\left(1 - \xi\right)$$





$$\phi_3 = (1-\xi)(1+\xi)$$