Solutions to Home Assignment 1

Warm-Up Exercises

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1. $$G_i = l_{\alpha\beta} M_{\alpha\beta} n_i$$

free index — takes on values 1, 2, 3 (i)

dummy indices — takes on values 1, 2 (α, β)

- The free index, i, indicates there are 3 equations.
- The two dummy indices, α and β, indicates a summation with 4 terms (2×2).

In "conventional" notation, G_i can be expressed as,

$$G_i = n_i \sum_{\alpha=1}^{2} \sum_{\beta=1}^{2} l_{\alpha\beta} M_{\alpha\beta}$$

Thus,

$$\boxed{\begin{array}{ll} i=1 : & G_1 = n_1 (l_{11}M_{11} + l_{12}M_{12} + l_{21}M_{21} + l_{22}M_{22}) \\ i=2 : & G_2 = n_2 (l_{11}M_{11} + l_{12}M_{12} + l_{21}M_{21} + l_{22}M_{22}) \\ i=3 : & G_3 = n_3 (l_{11}M_{11} + l_{12}M_{12} + l_{21}M_{21} + l_{22}M_{22}) \end{array}}$$

2. $$A_{ij} = Q_{ijkl} x_k x_l \quad \text{for } i=2, j=3$$

free indices. takes on values 1, 2, 3 (i, j)

dummy indices. takes on values 1, 2, 3 (k, l)

- The free indices, i and j, indicate there are 9 equations (3x3).
- The two dummy indices, k and ℓ, indicate a summation with 9 terms (3x3).

In "conventional" notation, A_{ij} can be expressed as,

$$A_{ij} = \sum_{k=1}^{3} \sum_{\ell=1}^{3} Q_{ijk\ell}\, \tau_k x_\ell$$

Here, we are interested in A_{23}. Thus,

$$\boxed{\begin{aligned} A_{23} = {} & Q_{2311}\tau_1 x_1 + Q_{2312}\tau_1 x_2 + Q_{2313}\tau_1 x_3 \\ & + Q_{2321}\tau_2 x_1 + Q_{2322}\tau_2 x_2 + Q_{2323}\tau_2 x_3 \\ & + Q_{2331}\tau_3 x_1 + Q_{2332}\tau_3 x_2 + Q_{2333}\tau_3 x_3 \end{aligned}}$$

3. $$a_{mn}\frac{\partial u_n}{\partial t} + f_m = 0$$

dummy index
takes on values 1, 2, 3

free index
takes on values 1, 2, 3

- The free index, m, indicates there are 3 equations.
- The dummy index, n, indicates a summation with 3 terms.

In "conventional" notation, the equation can be expressed as,

$$\sum_{n=1}^{3} a_{mn}\frac{\partial u_n}{\partial t} + f_m = 0$$

Thus,

$$\boxed{\begin{array}{ll} m=1: & a_{11}\frac{\partial u_1}{\partial t} + a_{12}\frac{\partial u_2}{\partial t} + a_{13}\frac{\partial u_3}{\partial t} + f_1 = 0 \\ m=2: & a_{21}\frac{\partial u_1}{\partial t} + a_{22}\frac{\partial u_2}{\partial t} + a_{23}\frac{\partial u_3}{\partial t} + f_2 = 0 \\ m=3: & a_{31}\frac{\partial u_1}{\partial t} + a_{32}\frac{\partial u_2}{\partial t} + a_{33}\frac{\partial u_3}{\partial t} + f_3 = 0 \end{array}}$$

4. $$E = \frac{1}{2}\sigma_{\underbrace{\alpha\beta}}\varepsilon_{\underbrace{\alpha\beta}}$$

dummy indices.
takes on values 1, 2

- The two dummy indices, α and β, indicate a summation with 4 terms (2×2).

In "conventional" notation, E can be expressed as,

$$E = \frac{1}{2}\sum_{\alpha=1}^{2}\sum_{\beta=1}^{2}\sigma_{\alpha\beta}\varepsilon_{\alpha\beta}$$

Thus,

$$\boxed{E = \frac{1}{2}\left(\sigma_{11}\varepsilon_{11} + \sigma_{12}\varepsilon_{12} + \sigma_{21}\varepsilon_{21} + \sigma_{22}\varepsilon_{22}\right)}$$

* Note : σ and ε are normally used to identify the stress tensor and the strain tensor, respectively, which are symmetric.

$$\Rightarrow \quad \sigma_{12} = \sigma_{21} \quad \text{and} \quad \varepsilon_{12} = \varepsilon_{21}$$

Using symmetry, E can also be expressed as

$$\boxed{E = \frac{1}{2}\left(\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + 2\sigma_{12}\varepsilon_{12} \right)}$$

5. $$\sigma_{23} = l_{2\hat{m}}\, l_{3\hat{n}}\, \underbrace{\sigma_{\hat{m}\hat{n}}}$$

dummy indices.
takes on values 1, 2, 3

- The dummy indices, m and n, indicates a summation with 9 terms (3x3)

In "conventional" notation, σ_{23} can be expressed as

$$\sigma_{23} = \sum_{m=1}^{3} \sum_{n=1}^{3} l_{2\hat{m}}\, l_{3\hat{n}}\, \sigma_{\hat{m}\hat{n}}$$

Thus,

$$\boxed{\begin{aligned} \sigma_{23} = \; & l_{2\hat{1}} l_{3\hat{1}} \sigma_{\hat{1}\hat{1}} + l_{2\hat{1}} l_{3\hat{2}} \sigma_{\hat{1}\hat{2}} + l_{2\hat{1}} l_{3\hat{3}} \sigma_{\hat{1}\hat{3}} \\ & + l_{2\hat{2}} l_{3\hat{1}} \sigma_{\hat{2}\hat{1}} + l_{2\hat{2}} l_{3\hat{2}} \sigma_{\hat{2}\hat{2}} + l_{2\hat{2}} l_{3\hat{3}} \sigma_{\hat{2}\hat{3}} \\ & + l_{2\hat{3}} l_{3\hat{1}} \sigma_{\hat{3}\hat{1}} + l_{2\hat{3}} l_{3\hat{2}} \sigma_{\hat{3}\hat{2}} + l_{2\hat{3}} l_{3\hat{3}} \sigma_{\hat{3}\hat{3}} \end{aligned}}$$

* Note : σ is normally used to identify the stress tensor, which is symmetric.

$$\Rightarrow \quad \sigma_{\bar{1}\bar{2}} = \sigma_{\bar{2}\bar{1}} \quad , \quad \sigma_{\bar{2}\bar{3}} = \sigma_{\bar{3}\bar{2}} \quad , \quad \sigma_{\bar{1}\bar{3}} = \sigma_{\bar{3}\bar{1}}$$

Using symmetry, σ_{23} can also be expressed as

$$\boxed{\begin{aligned} \sigma_{23} = \; & l_{2\bar{1}} l_{3\bar{1}} \sigma_{\bar{1}\bar{1}} + l_{2\bar{2}} l_{3\bar{2}} \sigma_{\bar{2}\bar{2}} + l_{2\bar{3}} l_{3\bar{3}} \sigma_{\bar{3}\bar{3}} \\ & + \left(l_{2\bar{1}} l_{3\bar{2}} + l_{2\bar{2}} l_{3\bar{1}} \right) \sigma_{\bar{1}\bar{2}} \\ & + \left(l_{2\bar{1}} l_{3\bar{3}} + l_{2\bar{3}} l_{3\bar{1}} \right) \sigma_{\bar{1}\bar{3}} \\ & + \left(l_{2\bar{2}} l_{3\bar{3}} + l_{2\bar{3}} l_{3\bar{2}} \right) \sigma_{\bar{2}\bar{3}} \end{aligned}}$$