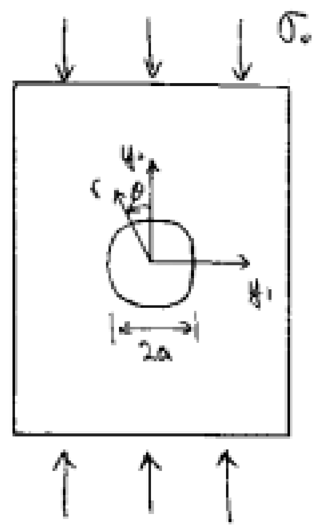
Solutions to Home Assignment #4

Warm-Up Exercises

1. To find the expressions for the stress components at the hole boundary as a function of θ, we will use the Airy stress functions. The problem we need to consider is an isotropic plate with a hole under compressive load.



From our notes (unit 8, p.20), our assumed stress function, ϕ, for an isotropic plate with a hole in polar coordinates is

$$\phi(r,\theta) = [A_0 + B_0 \ln r + C_0 r^2 + D_0 r^2 \ln r] + \left[A_2 r^2 + \frac{B_2}{r} + C_2 r^4 + D_2\right]\cos 2\theta \quad \text{—} \ ①$$

In order for the displacements to be single-valued, D_0 is set to equal to zero. For the stresses to be bounded as $r \to \infty$, C_2 also needs to be zero. The stress in polar coordinates can now be expressed as:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$= \frac{B_0}{r^2} + 2C_0 - \left[2A_2 + \frac{6B_2}{r^4} \right] \cos 2\theta \quad \text{—— ②}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$$

$$= -\frac{B_0}{r^2} + 2C_0 + \left[2A_2 + \frac{6B_2}{r^4} \right] \cos 2\theta \quad \text{—— ③}$$

$$\sigma_{r\theta} = -\frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta}$$

$$= \left[2A_2 - \frac{6B_2}{r^4} - \frac{2D_2}{r^2} \right] \sin 2\theta \quad \text{—— ④}$$

The boundary conditions are:

$\sigma_{rr} = \sigma_{r\theta} = 0$ @ $r = a$ ← stress-free at hole edge.

$\sigma_{yy} = \sigma_0$, $\sigma_{xy} = 0$ @ $y_2 \to \infty$

$\sigma_{xx} = 0$, $\sigma_{xy} = 0$ @ $y_1 \to \infty$

Using these boundary conditions, we can find the unknown constants in equations ② through ④. Skipping the math

unit8, p.26

(as described in notes), we can get

$$\sigma_{rr} = \frac{\sigma_0}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma_0}{2}\left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos 2\theta \quad \text{—— ⑤}$$

$$\sigma_{\theta\theta} = \frac{\sigma_0}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma_0}{2}\left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta \quad \text{—— ⑥}$$

$$\sigma_{r\theta} = -\frac{\sigma_0}{2}\left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right)\sin 2\theta \quad \text{—— ⑦}$$

At the hole boundary, i.e., $r=a$, the stresses are reduced to

$$\sigma_{rr} = 0 \quad \text{—— ⑧}$$

$$\sigma_{\theta\theta} = -\sigma_0\left(1 - 2\cos 2\theta\right) \quad \text{—— ⑨}$$

$$\sigma_{r\theta} = 0 \quad \text{—— ⑩}$$

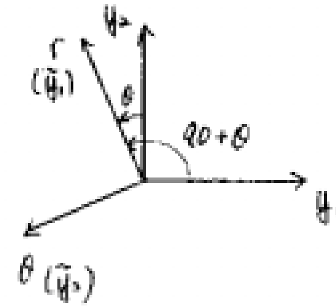
Now, we need to transform our stresses from the polar to Cartesian system.

Set

$$\tilde{\sigma}_{11} = \sigma_{rr} = 0$$

$$\tilde{\sigma}_{22} = \sigma_{\theta\theta}$$

$$\tilde{\sigma}_{12} = \sigma_{r\theta} = 0$$



To transform our r, θ (i.e. $\hat{y}_1 - \hat{y}_2$) system to our $y_1 - y_2$ system, we must rotate through an angle $-(90+\theta)$.

The transformation rule is

$$\sigma_{\alpha\beta} = l_{\alpha\sigma}\, l_{\beta\lambda}\, \hat{\sigma}_{\sigma\lambda} \qquad \text{--- (11)}$$

This equation reduces to

$$\sigma_{11} = l_{1\tilde{2}}\, l_{1\tilde{2}}\, \tilde{\sigma}_{22} = \cos^2\theta\ \sigma_{\theta\theta} \qquad \text{--- (12)}$$

$$\sigma_{22} = l_{2\tilde{2}}\, l_{2\tilde{2}}\, \hat{\sigma}_{22} = \cos^2(90+\theta)\, \sigma_{\theta\theta} = \sin^2\theta\, \sigma_{\theta\theta} \qquad \text{--- (13)}$$

$$\sigma_{12} = l_{1\tilde{2}}\, l_{2\tilde{2}}\, \tilde{\sigma}_{22} = (-\cos\theta)(-\sin\theta)\, \sigma_{\theta\theta} = \sin\theta\cos\theta\, \sigma_{\theta\theta} \qquad \text{--- (14)}$$

& $l_{1\tilde{2}} = \cos(-180-\theta) = -\cos\theta$

$l_{2\tilde{2}} = \cos(-90-\theta) = -\sin\theta$

Therefore, plugging the expression for $\sigma_{\theta\theta}$ in equations (13) through (14), we get:

$$\boxed{\begin{aligned} \sigma_{11} &= -\sigma_0 (1-2\cos 2\theta)\cos^2\theta \\ \sigma_{22} &= -\sigma_0 (1-2\cos 2\theta)\sin^2\theta \\ \sigma_{12} &= \sigma_0 (1-2\cos 2\theta)\sin\theta\cos\theta \end{aligned}}$$

For use in problems 2 and 3, the stress can be expressed in normalized form as:

y_1-y_2 coordinate:

$$\sigma_{11}/\sigma_0 = -(1-2\cos 2\theta)\cos^2\theta$$

$$\sigma_{22}/\sigma_0 = -(1-2\cos 2\theta)\sin^2\theta$$

$$\sigma_{12}/\sigma_0 = (1-2\cos 2\theta)\sin\theta\cos\theta$$

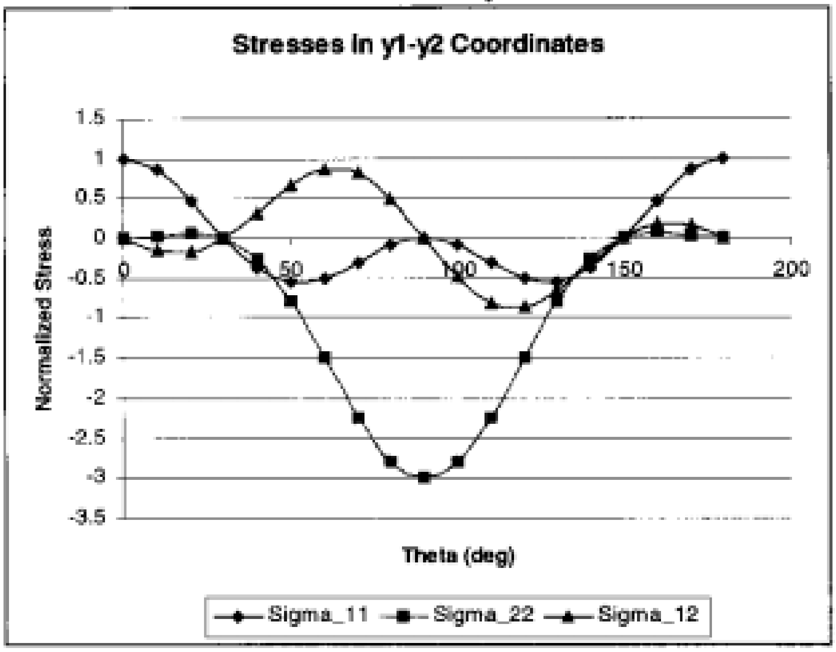
r-θ coordinate:

$$\sigma_{rr}/\sigma_0 = 0$$

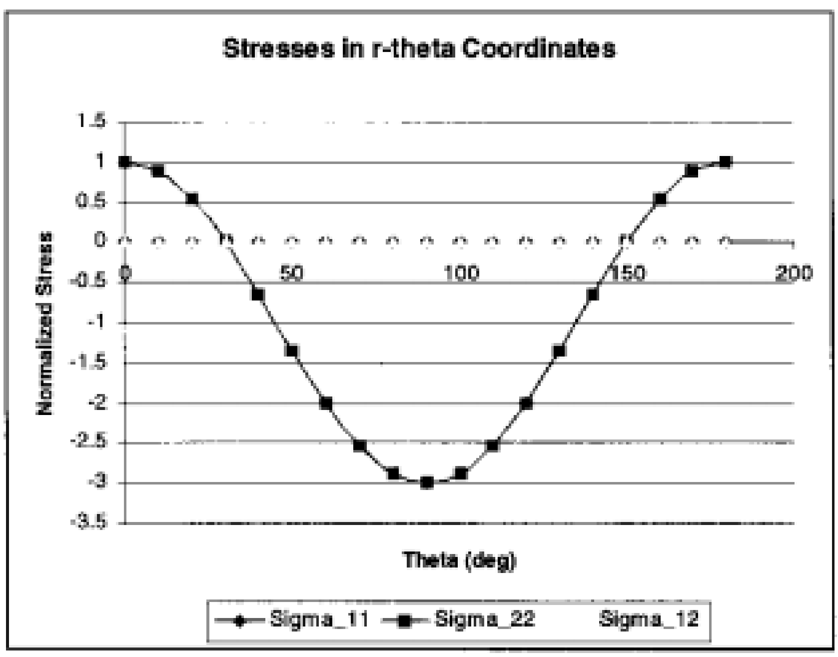
$$\sigma_{\theta\theta}/\sigma_0 = -(1-2\cos 2\theta)$$

$$\sigma_{r\theta}/\sigma_0 = 0$$

2. Stresses in y_1-y_2 coordinate system



3. Stresses in r-θ coordinate system



4. We can derive a number of interesting results from these plots.

   ① Extensional stresses (σ_{11} , σ_{22} , $\sigma_{\theta\theta}$) are symmetric about each 90° rotation.

   ② Shear stresses (σ_{12}) are anti-symmetric about each 90° rotation.

   ③ All stresses go to zero at $\theta = 30°, 150°, 210°$ and $330°$

   ④ σ_{rr} and $\sigma_{r\theta}$ are zero at hole edge. Only $\sigma_{\theta\theta}$ is non-zero.

   ⑤ In the plots, all stresses are normalized by σ_0, which is negative in the present case. So, the actual stress state has the same shape but opposite sign.