Chapter 3

Integral Boundary Layer Equations for Three-Dimensional Flows

3.1 Definitions

The three-dimensional integral boundary layer equations derived using a Cartesian coordinate system are:

$$\frac{\delta}{C_\tau}\frac{\partial C_\tau}{\partial \xi} = K_c\left(C_{\tau\,eq}^{1/2} - C_\tau^{1/2}\right) \tag{3.1}$$

$$\frac{\partial}{\partial x}\left(\rho_e q_e^3 \theta_x^*\right) + \frac{\partial}{\partial z}\left(\rho_e q_e^3 \theta_z^*\right) + \rho_e q_e^2 \delta_x^{**}\frac{\partial u_e}{\partial x} + \rho_e q_e^2 \delta_z^{**}\frac{\partial w_e}{\partial z} = 2D \tag{3.2}$$

$$\frac{\partial}{\partial x}\left(\rho_e q_e^2 \theta_{xx}\right) + \frac{\partial}{\partial z}\left(\rho_e q_e^2 \theta_{xz}\right) + \rho_e q_e \delta_x^*\frac{\partial u_e}{\partial x} + \rho_e q_e \delta_z^*\frac{\partial u_e}{\partial z} = \tau_{xw} \tag{3.3}$$

$$\frac{\partial}{\partial x}\left(\rho_e q_e^2 \theta_{zx}\right) + \frac{\partial}{\partial z}\left(\rho_e q_e^2 \theta_{zz}\right) + \rho_e q_e \delta_z^*\frac{\partial w_e}{\partial z} + \rho_e q_e \delta_z^*\frac{\partial w_e}{\partial z} = \tau_{zw} \tag{3.4}$$

Equation 3.1 is the two-dimensional turbulent shear stress lag equation, 3.2 is the kinetic energy equation, and 3.3 and 3.4 are the x and z momentum equations. The origin of equation 3.1 is given in reference [21] and the derivation of equations 3.2-3.4 may be found in [38]. K_c is an empirical constant set to 5.6. ξ is the lag direction which is taken to be the chordwise direction, and the x-z coordinate system is an arbitrary, local, 2-D surface coordinate system in a plane tangent to the 3-D BL surface. The main assumptions in these equations are that pressure is constant in the normal direction through the thickness of the boundary layer, and that only diffusion normal to the wall is significant.

In Equations 3.1 to 3.4, δ^*'s denote displacement thicknesses, θ's signify momentum thicknesses, θ^*'s represent energy thicknesses, and δ^{**}'s are density thicknesses. In addition, $C_\tau^{1/2}$ is the shear stress coefficient, $C_\tau^{1/2}{}_{eq}$ is the equilibrium shear stress coefficient, τ's are wall shear stresses, and D is the turbulent dissipation function. Section §B.1 provides the definitions of the thicknesses for a streamwise-crossflow coordinate system (called the 1-2 coordinate system).

These integral boundary layer equations have a physical interpretation. Equation 3.2 may be thought of as a divergence of kinetic energy deficit (the $\rho_e q_e^3 \theta_x^*$ and $\rho_e q_e^3 \theta_z^*$ terms) balanced by mechanical work deficit ($\rho_e q_e^2 \delta_x^{**} \frac{\partial u_e}{\partial x}$, $\rho_e q_e^2 \delta_z^{**} \frac{\partial w_e}{\partial z}$) and dissipation ($D$). The mechanical work terms are the products of pressure forces and densities; the pressures may be recognized using Euler's Relation $dp = -\rho\, q\, dq$; and the densities may be seen by the definitions of $q_e \delta_x^{**}, q_e \delta_z^{**}$. Equation 3.3 contains the x-momentum deficit due to fluxes in the x-direction ($\rho_e q_e^2 \theta_{xx}$) and x-momentum deficit due to fluxes in the z-direction ($\rho_e q_e^2 \theta_{xz}$) balanced by pressure gradient term ($\rho_e q_e \delta_x^* \frac{\partial u_e}{\partial x} + \rho_e q_e \delta_z^* \frac{\partial u_e}{\partial z}$) and wall shear stresses (τ_{xw}). The pressure gradient term is seen once again using Euler's Relation. Equation 3.4 is a similar expression for z-momentum deficit.

3.2 Boundary Layer Domain

The three-dimensional boundary layer equations are a set of hyperbolic partial differential equations [59, 11]. Consequently, these equations require an initial or starting solution, and boundary conditions which depend on the mathematical characteristics entering or leaving the domain. A typical boundary layer domain is depicted in Figure 3.2. The starting solution for Equations 3.2 to 3.4 is specified at the attachment line near the wing leading edge, and the starting solution for Equation 3.1 is specified at the transition line. Boundary conditions are applied along the symmetry line, and the wingtip and waketip boundaries.

The main purpose of this research is to implement Fully Simultaneous coupling in three dimensions which is a difficult problem in itself. Therefore more complicated issues such as corner flows (at wing/body junctures) and three-dimensional free transition will not be