2.2) A) Stress - strain rate relation for a Newt. fluid

B) Viscosity coefficient

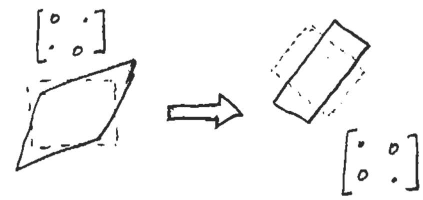
C) Vorticity and circulation

White 23-29, 59-69, 89-91
Knethe & Chow 40-50,
Batch - 71-99.

A) Stress - Strain rate relation

For any stress tensor σ_{ij} in coordinates x_i, a set of principle axis exists such that

$$\sigma'_{ij} = \begin{bmatrix} \sigma'_{11} & 0 & 0 \\ 0 & \sigma'_{22} & 0 \\ 0 & 0 & \sigma'_{33} \end{bmatrix}$$



$\sigma'_{11}, \sigma'_{22}, \sigma'_{33}$ are eigenvalues of σ_{ij} (σ'_{ij}) invariant under coord. transformation

Physical significance is the avg. principal stress

$$1/3 \left(\sigma'_{11} + \sigma'_{22} + \sigma'_{33}\right) \equiv 1/3\, \sigma'_{ii} = -p \quad \text{(static pressure)}$$

Note: This is the only stress in a fluid at rest ($\sigma_{ij} = -p\,\delta_{ij}$) (and generally in an inviscid fluid)

This suggest that

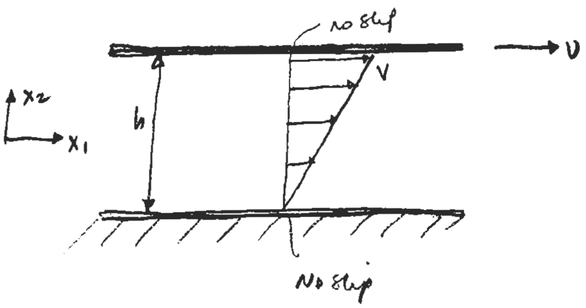
$$\sigma_{ij} = \underbrace{d_{ij}}_{\substack{\text{deviatoric}\\ \text{stresses}\\ \text{(non-isotropic)}}} - \underbrace{p\,\delta_{ij}}_{\substack{\text{spherical stresses}\\ \text{(isotropic)}}} = (\sigma_{ij} + p\,\delta_{ij}) + (-p\,\delta_{ij})$$

such that $d_{ij} = 0$ for a fluid at rest

By definition of a fluid, d_{ij} must depend on the velocity field, more precisely, on strain rate tensor.

$$d_{ij} = f(e_{ij})$$

For a Newtonian fluid this relationship is linear. Consider two plates in motion



$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\frac{du_1}{dx_2}$$

$$\sigma_{12} \sim e_{12} \qquad \sigma_{12} = \mu\frac{V}{h} = \mu\frac{du_1}{dx_2} = 2\mu e_{12}$$

μ is the coefficient of viscosity

In general,

$$d_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} \quad ; \quad e_{kk} = e_{11} + e_{22} + e_{33} = \nabla\cdot\vec{u}$$

λ is the coefficient of bulk viscosity (e_{kk} – normal stresses)

$$\therefore \sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} + \lambda\nabla\cdot\vec{u}\,\delta_{ij}$$

$$\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Since

$$p = -\tfrac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) \implies \lambda = -\tfrac{2}{3}\mu \quad \text{– Stoke's Hypothesis}$$

C) ~~Vorticity~~

Finally we can write a single deformation law for a Newtonian (linear) viscous fluid as:

$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu\left(e_{ij} - \tfrac{1}{3}\nabla\cdot\vec{u}\,\delta_{ij}\right)$$

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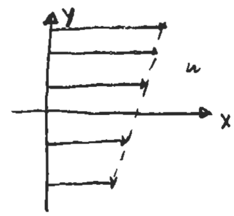
$$\rho\frac{Du_i}{Dt} = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]$$

If ρ = const, μ = const $\rightarrow$ $D\vec{u}/Dt = \vec{f} - \nabla p/\rho + \nu\left[\nabla^2\vec{u} + \nabla(\cancelto{0}{\nabla\cdot\vec{u}})\right]$ $\quad \nu = \mu/\rho$

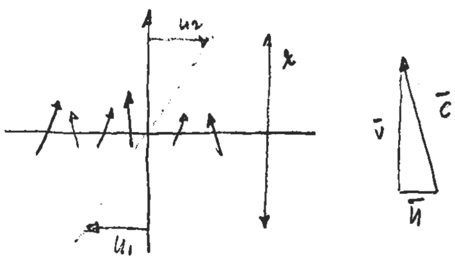
B) Molecular Basis for viscosity and pressure.

Consider mean shear flow



$$u(y) = u_0 + \frac{du}{dy}\cdot y$$

Mass and momentum transport across $y=0$



upward moving molecules

λ = mean free path

$\bar{c}$ = mean molecular speed

1) Mass flux $= \dot{m} = \rho\bar{v} \simeq \rho\bar{c}$; ρ = molecular mass density

2) x- momentum flux $= \dot{m}\bar{u} \simeq -\rho\bar{c}\,\frac{\lambda}{2}\frac{du}{dy}$ $\qquad \bar{u} \simeq u_1 = -\frac{du}{dy}\cdot\lambda/2$

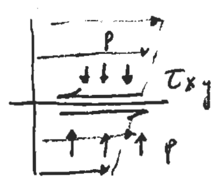
3) y - momentum flux $= \dot{m}\bar{v} = \rho\bar{c}^2$

suggest $\mu = \frac{1}{2}\rho\bar{c}\lambda$

At length scales $\gg \lambda$

x-mom. flux = tang stress $= -\mu\frac{du}{dy} = \tau_{xy}$

y-mom flux = normal stress $= p$



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c) Vorticity and Circulation

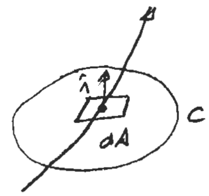
$$\text{Vorticity} = \vec{\omega} = \nabla \times \vec{u} \qquad \text{curl of velocity field}$$

$$\nabla \times \vec{u} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$$

Stokes theorem

$$\iint_A (\nabla \times \vec{u}) \cdot \hat{n}\, dA = \oint_C \vec{u} \cdot d\vec{l}$$

$$\iint_A \vec{\omega} \cdot \hat{n}\, dA = \oint_C \vec{u} \cdot d\vec{l}$$



Example: rotating cylinder of fluid

$$\omega A = 2\pi r\, v_{tang}$$

$$\omega \pi r^2 = 2\pi r\, \Omega r$$

$$\underline{\underline{\omega = 2\Omega}} \quad - \quad \text{Vorticity is } 2\times \text{ angular velocity}$$

To understand vorticity changes in a flow

$$\frac{D\vec{u}}{Dt} = \vec{f}_{body} - \frac{\nabla p}{\rho} + \nu \nabla^2 \vec{u} \qquad (\text{incomp}, \ \mu = \text{const})$$

$$\nu = \mu/\rho$$

$$\vec{f}_{body} = \nabla \Omega \quad - \text{ conservative body force}$$

Vector identity:

$$\vec{u} \cdot \nabla \vec{u} = \nabla (u^2/2) - \vec{u} \times \vec{\omega}$$

irrotational flow

$$\nabla \times \nabla \phi = 0$$

and,

$$\nabla \times \nabla (\ ) = 0 \quad , \quad \nabla \cdot (\nabla \times (\ )) = 0$$

$$\nabla \times \left[\frac{\partial \vec{u}}{\partial t} + \nabla (u^2/2) - \vec{u} \times \vec{\omega} = \nabla \Omega - \frac{1}{\rho} \nabla p + \nu \nabla^2 \vec{u} \right]$$

$$\Rightarrow \quad \frac{\partial \vec{\omega}}{\partial t} + \nabla \times \cancelto{0}{\nabla}(u^2/2) - \nabla \times (\vec{u} \times \vec{\omega}) = \nabla \times \cancelto{0}{\nabla}\Omega - \frac{1}{\rho} \nabla \times \cancelto{0}{\nabla} p$$
$$+ \nu \nabla \times \nabla^2 \vec{u}$$

$$\nabla \times (\vec{u} \times \vec{\omega}) = \vec{u}\,(\cancelto{0}{\nabla \cdot \vec{\omega}}) + \vec{\omega} \cdot \nabla \vec{u} - \vec{\omega}\,\cancelto{0}{\nabla \cdot \vec{u}} - \vec{u} \cdot \nabla \vec{\omega}$$

$$\nabla \times (a \times b) = \vec{a}\,(\nabla \cdot \vec{b}) + \vec{b} \cdot \nabla \vec{a} - \vec{b}\,(\nabla \cdot \vec{a}) - \vec{a} \cdot \nabla \vec{b}$$

$$\Rightarrow \quad \frac{D\vec{\omega}}{Dt} = \underbrace{\vec{\omega} \cdot \nabla \vec{u}}_{\text{vorticity stretching + tilting term}} + \underbrace{\nu \nabla^2 \vec{\omega}}_{\text{vorticity diffusion}}$$

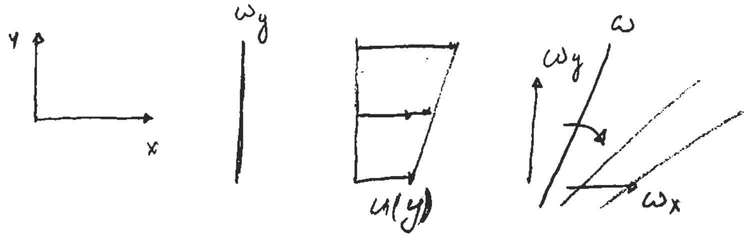
Evolution of vorticity is unaffected by the pressure field. Purely kinematic quantity, just motion.

Consider x component of vorticity equation

$$\frac{D\omega_x}{Dt} = \underbrace{\omega_x \frac{\partial u}{\partial x}}_{\text{stretching term}} + \underbrace{\omega_y \frac{\partial u}{\partial y} + \omega_z \frac{\partial u}{\partial z}}_{\text{tipping or tilting term}}$$

$$\frac{1}{\Delta} \frac{\partial u_x}{\partial x} \cdot \Delta x$$

$$\frac{1}{\omega_x} \frac{D\omega_x}{Dt} = \frac{\partial u}{\partial x} \quad \leftarrow \text{vortex stretching}$$



due to $\frac{\partial u}{\partial y}$

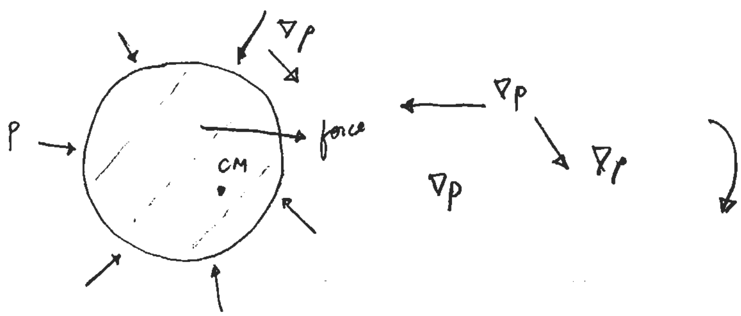
$\vec{\omega}$ gets tipped / tilted in x direction – rate of creation of x vorticity. Ill'ly for z component

In a 2-D flow $\quad \vec{\omega} = (0, 0, \omega_z)$

$$\Rightarrow \quad \frac{D\omega_z}{Dt} = \frac{D\omega}{Dt} = 0$$

Compressible, inviscid form

$$\frac{D}{Dt}\left(\vec{\omega}/\rho\right) = \left(\frac{\vec{\omega}}{\rho}\right)\cdot\nabla\vec{u} \quad + \quad \underbrace{\frac{\nabla p}{\rho} \times \nabla(1/\rho)}$$



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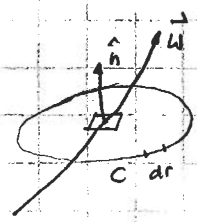
Circulation

$$\Gamma_c = \oint_c \vec{u} \cdot d\vec{r} \quad / \quad \oint \nabla \times \vec{u} \cdot \hat{n}\, dA$$



$$= \iint (\vec{\omega} \cdot \hat{n})\, dA$$

vorticity

$$\frac{D\Gamma_c}{Dt} = \frac{D}{Dt} \oint \vec{u} \cdot d\vec{l}$$

If we consider a contour with same fluid particles

$$\frac{D\Gamma_c}{Dt} = \oint_c \frac{D}{Dt}(\vec{u} \cdot d\vec{l})$$

$$= \oint_c \frac{D\vec{u}}{Dt} \cdot d\vec{l} + \oint_c \vec{u} \cdot \frac{D d\vec{l}}{Dt}$$

Rate of change of $d\vec{l}$ is $\frac{\delta (d\vec{l})}{\delta t}$ or $d\vec{U}$

so $\frac{D\, d\vec{l}}{Dt} = d\vec{U}$

Hence $$\oint_c \vec{u} \cdot \frac{D\, d\vec{l}}{Dt} = \oint_c \vec{u} \cdot d\vec{u} = \oint_c d\left(\frac{\vec{u} \cdot \vec{u}}{2}\right) = 0$$

exact differential

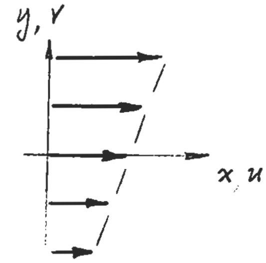
$$\therefore \quad \frac{D\Gamma_c}{Dt} = \oint \frac{D\vec{u}}{Dt} \cdot d\vec{l}$$

$$= \oint \left[-\frac{\nabla p}{\rho} + \vec{f}_{body} + \nabla \cdot \bar{\bar{\sigma}} \right] \cdot d\vec{l} \quad \text{(Kelvin's Theorem)}$$

Uniform density, inviscid, conservative f_{body}

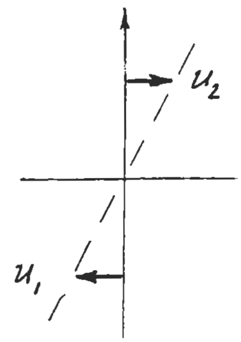
$\Rightarrow \frac{D\Gamma_c}{Dt} = 0$ If $\Gamma_c = 0$ for a fluid contour, it always has $\Gamma_c = 0$

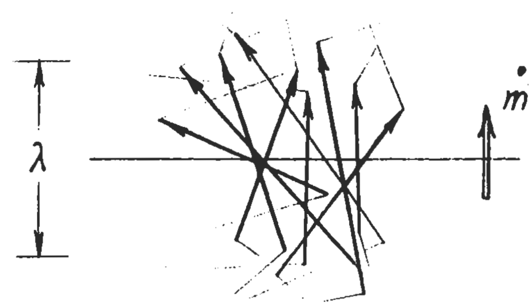
Molecular Basis for Viscosity and Pressure



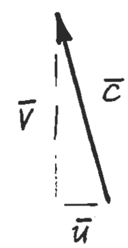
Consider mean shear flow: $u(y) = u_0 + \frac{du}{dy}\, y$

Examine mass and momentum transport across $y=0$ plane in frame moving at $u = u_0$





upward-moving molecules crossing $y=0$ plane



velocity of average upward-moving molecule

λ = mean free path

$\bar{c}$ = mean molecular speed

$\bar{u} \simeq u_1 = -\frac{du}{dy}\frac{\lambda}{2}$

$\bar{v} \simeq \bar{c}$

Effects of upward moving molecules on space above $y=0$ plane:

- Mass flux $= \dot{m} = \rho\bar{v} \simeq \rho\bar{c}$; ρ = molecular mass density
- x–Momentum flux $= \dot{m}\bar{u} \simeq -\rho\bar{c}\,\frac{\lambda}{2}\frac{du}{dy}$
- y–Momentum flux $= \dot{m}\bar{v} \simeq \rho\bar{c}^{\,2}$

suggests $\mu \simeq \frac{1}{2}\rho\bar{c}\lambda$

actually, $\mu = 0.499\,\rho\bar{c}\lambda$

In length scales $\gg \lambda$:

x-mom. flux = tangential stress on $y=0$ plane $= -\mu\frac{du}{dy} \equiv \tau_{xy}$

y-mom. flux = normal " " " " $= p$

By considering downward-moving molecules, we get mass and momentum flux into space below $y=0$ plane