2-1 Conservation Laws.

A) Mass
B) Stress Tensor, Fluxes - momentum
C) Conservation of Momentum

Batch 73-75, 137-151
Schl 47-61
White 61-65

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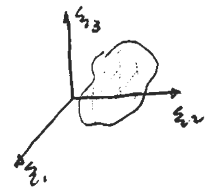
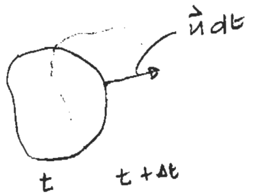
A). Conservation of Mass.

Mass in a C.V : $M(t) = \iiint_V \rho \, dV$ $dV = d\xi_1 \, d\xi_2 \, d\xi_3$



Lagrangian View



assert $\left.\frac{dm}{dt}\right|_\xi = 0$

$$\frac{d}{dt}\iiint \rho \, dV = \iiint \left[\left.\frac{\partial \rho}{\partial t}\right|_\xi dV + \rho \left.\frac{\partial V}{\partial t}\right|_\xi \right] = 0$$

$$= \iiint \left[\frac{D\rho}{Dt} + \rho \nabla\cdot\vec{u} \right] dV = 0$$ must hold for any C.V

(last lect $\frac{1}{V}\frac{dV}{dt} = \nabla\cdot\vec{u}$)

Example:

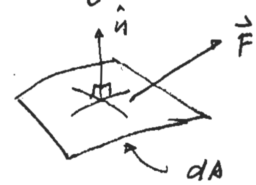
is invalid

=> Must have everywhere : $\boxed{\frac{D\rho}{Dt} + \rho \nabla\cdot\vec{u} = 0}$ - convective form of continuity equation

Eulerian View

flux of any field quantity $\vec{F}(\vec{x}, t)$ through a C·V

$$= \vec{F} \cdot \hat{n}$$



$$d\vec{A} = \hat{n}\, dA$$

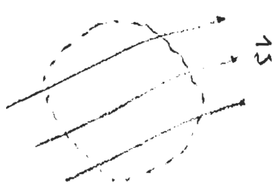
Gauss's Theorem

$$\oiint_{C \cdot V} \vec{F} \cdot \hat{n}\, dA = \iiint_V \nabla \cdot \vec{F}\, dV$$

Introduce <u>mass flux</u> $= \rho \vec{u} \cdot \hat{n}$

mass flow $= \rho \vec{u} \cdot d\vec{A}$

Now C·V fixed in space



mass in C.V. is $M(t) = \iiint_V \rho\, dV$

where $dV = dx_1\, dx_2\, dx_3$

assert $\dfrac{dM}{dt} =$ (local change) + (mass flow in − mass flow out) $= 0$

$$\frac{dM}{dt} = \frac{d}{dt} \iiint_V \rho\, dV \;\cancel{\text{and}}\; + \oiint_{C \cdot V} \rho \vec{u} \cdot \hat{n}\, dA = 0$$

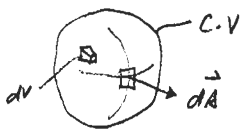
$$\iiint_V \left.\frac{\partial \rho}{\partial t}\right|_x dV + \rho \left.\cancel{\frac{\partial\, dV}{\partial t}}^{\,0}\right|_x = + \iiint_V \nabla \cdot \rho \vec{u}\, dV \quad \leftarrow \text{Gauss Theorem}$$

$$\Rightarrow \iiint_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} \right] dV = 0$$

Conservative or Divergence form.

B) Momentum Conservation

Lagrangian View



momentum in C.V : $\vec{M} = \iiint_V \rho \vec{u}\, dV$

$dV = d\xi_1\, d\xi_2\, d\xi_3$

assert $\dfrac{d\vec{M}}{dt} = \sum$ forces on C.V

$$\frac{d}{dt}\iiint_V \rho \vec{u}\, dV = F_{BODY} + F_{surface}$$

Example

$F_{body} = -g\hat{z}$ gravity

$= \Omega^2 \vec{r}$ centrifugal

$= 2\vec{\Omega}\times\vec{u}$ coriolis

$$\frac{d}{dt}\iiint_V \rho\vec{u}\, dV = \iiint_V \left[\frac{\partial \vec{u}}{\partial t}\Big|_\xi \rho\, dV + \underbrace{\vec{u}\,\frac{\partial}{\partial t}(\rho\, dV)\Big|_\xi}_{=0 \text{ conservation of mass}}\right] = F_{body} + F_{surface}$$

$$\therefore \iiint_V \rho\,\frac{\partial \vec{u}}{\partial t}\Big|_\xi dV = \iiint_V \rho \vec{f}_B\, dV + \underbrace{\oint\!\!\oint \bar{\bar{\sigma}}\cdot d\vec{A}}_{\iiint \nabla\cdot\bar{\bar{\sigma}}\, dV \ (\text{Gauss})}$$

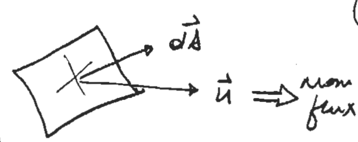
$$\Rightarrow \boxed{\rho\,\frac{D\vec{u}}{Dt} = \rho\vec{f}_{body} + \nabla\cdot\bar{\bar{\sigma}}}$$ convective form of mom eqn.

$\bar{\bar{\sigma}}$ is the stress tensor $[3\times 3]$

For inviscid flow $\bar{\bar{\sigma}} = -p\bar{\bar{I}}$ or $\sigma_{ij} = -p\,\delta_{ij}$ $\quad \bar{\bar{\sigma}} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$

$$\Rightarrow \frac{D\vec{u}}{Dt} = \vec{f} - \frac{\nabla p}{\rho}$$ – (Euler Equation)

Eulerian View



Introduce $\overrightarrow{\text{momentum flux}} = (\text{mass flux}) \cdot \dfrac{\overrightarrow{\text{momentum}}}{\text{unit mass}}$

$$= (\rho\vec{u}\cdot\hat{n})\,\vec{u}$$

$$M(t) = \iiint_V \rho\vec{u}\, dV$$

assert $\dfrac{d\vec{M}}{dt} = \sum \overrightarrow{\text{forces}}$ on C.V $+$ $\overrightarrow{\text{mom. flow in}}$ $-$ $\overrightarrow{\text{mom flow out}}$

$$\frac{d}{dt}\iiint_V \rho\vec{u}\, dV = \iiint_V \rho\vec{f}\, dV + \oiint_S \bar{\bar{\sigma}}\cdot d\vec{A} - \oiint_{C.V} (\rho\vec{u}\cdot\hat{n})\,\vec{u}\, dA$$

$$\iiint_V \left.\frac{\partial \rho\vec{u}}{\partial t}\right|_x dV + \rho\vec{u}\,\cancelto{0 \text{ fixed C.V}}{\left.\frac{\partial}{\partial t}(dV)\right|_x} = \iiint_V \rho\vec{f}\, dV + \iiint_V \nabla\cdot\bar{\bar{\sigma}}\, dV - \iiint_V \nabla\cdot[(\rho\vec{u})\,\vec{u}]\, dV$$

$$\Rightarrow \quad \frac{\partial \rho\vec{u}}{\partial t} + \nabla\cdot\left[(\rho\vec{u})\,\vec{u}\right] = \rho\vec{f}_b + \nabla\cdot\bar{\bar{\sigma}}$$ divergence or conservative form

where, in cartesian coord system

$$\nabla\cdot\left[(\rho\vec{u})\,\vec{u}\right] = \frac{\partial}{\partial x_j}(\rho u_j u_i) = \frac{\partial}{\partial x}(\rho u\vec{u}) + \frac{\partial}{\partial y}(\rho v\vec{u}) + \frac{\partial}{\partial z}(\rho w\vec{u})$$

$$= \hat{i}\,\nabla\cdot(\rho\vec{u}\,u) + \hat{j}\,\nabla\cdot(\rho\vec{u}\,v) + \hat{k}\,\nabla\cdot(\rho\vec{u}\,w)$$

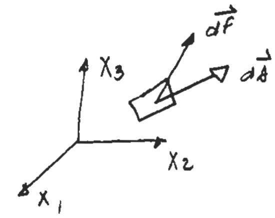
We must still specify $\bar{\bar{\sigma}}$ to close the system.

Stress Tensor

Surface dF_i acting on an area element dA_i is given by

$$dF_i = \sigma_{ij}\, dA_j$$

↑ stress tensor defn.



σ_{ij} is like a vector operator which converts an area dA_j into a force dF_j acting on that area.

$$\sigma_{ij} = \frac{\text{force in direction } i}{\text{per unit area in direction } j}$$

σ_{ij} - stress in j direction on plane normal to i

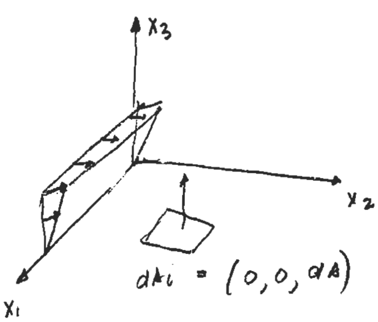
In vector form

$$\begin{Bmatrix} dF_1 \\ dF_2 \\ dF_3 \end{Bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \cdot & \sigma_{22} & \cdot \\ \cdot & \cdot & \sigma_{33} \end{bmatrix} \begin{Bmatrix} dA_1 \\ dA_2 \\ dA_3 \end{Bmatrix}.$$

(*) $dF_1 = \ldots$

Example

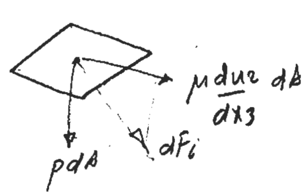
Shear flow over a wall



$$u_i = \left(0,\ \frac{du_2}{dx_3} x_3,\ 0\right)$$

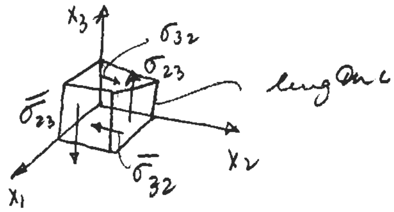
$$\sigma_{ij} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & \mu \frac{\partial u_2}{\partial x_3} \\ 0 & \mu \frac{\partial u_2}{\partial x_3} & -p \end{bmatrix}$$

$$\Rightarrow dF_i = \left(0,\ \mu \frac{\partial u_2}{\partial x_3} dA,\ -p\, dA\right).$$



Symmetry is an important property of the stress tensor

Consider cube with surface stresses.



acceleration

$$a_3 = \frac{\Sigma F_3}{m} = \frac{l^2(\sigma_{23} - \bar{\sigma}_{23})}{\rho l^3}$$

$$= \frac{\sigma_{23} - \bar{\sigma}_{23}}{\rho l}$$ as $l \to 0$, $a_3 \to \infty$ unless $\underline{\underline{\sigma_{23} = \bar{\sigma}_{23}}}$

angular acceleration

$$\alpha_1 = \frac{\Sigma M_1}{I} \sim \frac{l^3(\sigma_{23} - \sigma_{32})}{\rho l^4}$$, as $l \to 0$, $\alpha_1 \to \infty$ unless $\underline{\underline{\sigma_{23} = \sigma_{32}}}$.

Hence σ_{ij} is symmetric

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$$dF_1 = \sigma_{11}\, dA_1 + \sigma_{12}\, dA_2 + \sigma_{13}\, dA_3$$

$$= \sigma_{11}\, dx_2\, dx_3 + \sigma_{12}\, dx_1\, dx_3 + \sigma_{13}\, dx_1\, dx_2$$

$\underline{\underline{\text{ERASE}}}$ $\underline{\underline{\text{BOARD}}}$

$$\sigma_{11} = \bar{\sigma}_{11} + \frac{\partial \sigma_{11}}{\partial x_1}\, dx_1$$

$$\therefore\ dF_{1\,net} = \frac{\partial \sigma_{11}}{\partial x_1} \cdot dx_1\, dx_2\, dx_3 + \frac{\partial \sigma_{12}}{\partial x_2}\, dx_2\, dx_1\, dx_3 + \frac{\partial \sigma_{13}}{\partial x_3}\, dx_3\, dx_1\, dx_2$$

$$\Rightarrow \quad f_1 = \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}$$ Illy for other components given $\vec{f}_{surf} = \nabla \cdot \bar{\bar{\sigma}}_{ij}$ //.