1) a) $\nabla\left(\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y} = 0\right)$

$$\Rightarrow \frac{\partial \nabla s}{\partial t} + u\frac{\partial \nabla s}{\partial x} + v\frac{\partial \nabla s}{\partial y} + \nabla u\frac{\partial s}{\partial x} + \nabla v\frac{\partial s}{\partial y} = 0$$

or $$\frac{D\nabla s}{Dt} = -\nabla u\frac{\partial s}{\partial x} - \nabla v\frac{\partial s}{\partial y} = -\nabla u\, s_x - \nabla v\, s_y$$

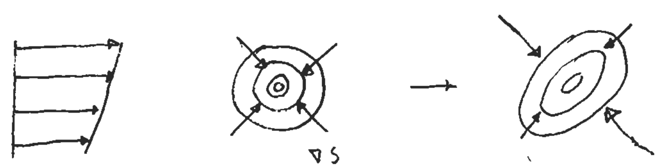
$$= -\nabla\vec{u}\cdot\nabla s + \bar{\bar{\omega}}\cdot\nabla s \qquad \text{where } \nabla\vec{u} = \tfrac{1}{2}\bar{\bar{\omega}} + \bar{\bar{e}}$$

or $$\frac{D}{Dt}\begin{Bmatrix} s_x \\ s_y\end{Bmatrix} = -\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y}\end{bmatrix}\begin{Bmatrix} s_x \\ s_y\end{Bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) \\ -\frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & 0\end{bmatrix}\begin{Bmatrix} s_x \\ s_y\end{Bmatrix} - \begin{bmatrix}\frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y}\end{bmatrix}\begin{Bmatrix} s_x \\ s_y\end{Bmatrix}$$

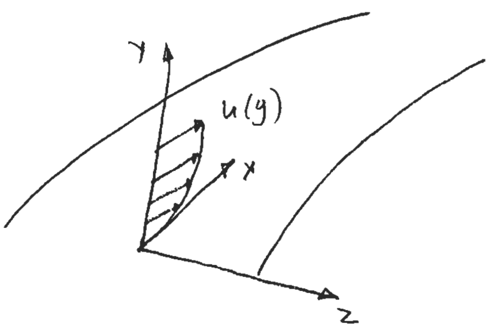
$$= \underbrace{\tfrac{1}{2}\bar{\bar{\omega}}\cdot\nabla s}_{\text{tilting}} - \underbrace{\bar{\bar{e}}\cdot\nabla s}_{\text{stretching}}$$

b) $\vec{u} = Ky\hat{i}$

$\therefore\ \vec{\omega} = -K\hat{k}$ (rotation) $\qquad \bar{\bar{e}} = \frac{1}{2}\begin{bmatrix} 0 & K \\ K & 0\end{bmatrix}$ (strain) $\qquad \Rightarrow \frac{D\nabla s}{Dt} = -\frac{1}{2}K\left(\hat{k}\times\nabla s\right) - \frac{1}{2}K\left(s_y\hat{i} + s_x\hat{j}\right)$



Direction and magnitude of ∇s changes.

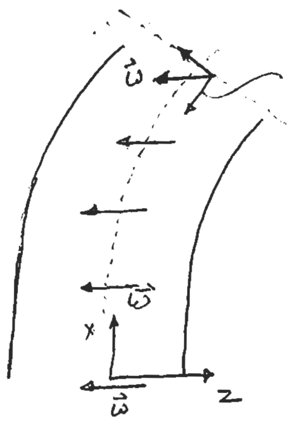


$$\vec{\omega} = 0\hat{i} + 0\hat{j} + \omega_z \hat{k} \quad ; \quad \omega_z = -\frac{\partial u}{\partial y} \quad (\nabla \times \vec{u})$$

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla)\vec{u} = \omega_z \frac{\partial}{\partial z}(\vec{u}) \simeq 0$$

$\therefore$ $\vec{\omega}$ is nearly constant along streamlines ($\vec{u} \cdot \nabla \omega_z \simeq 0$)

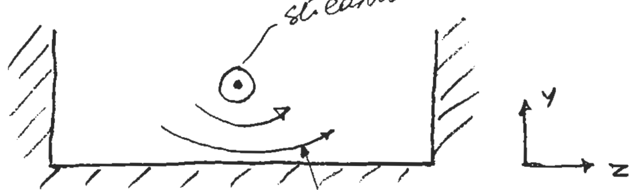
Top View:



streamwise vorticity appears.

Looking downstream:

streamwise vorticity



inward crossflow appears.

3)

$$X' = x - c_x t \qquad Y' = y - c_y t \qquad t' = t$$

$$\Longrightarrow \quad \frac{\partial(\ )}{\partial x} = \frac{\partial(\ )}{\partial x'}$$

$$\frac{\partial(\ )}{\partial y} = \frac{\partial(\ )}{\partial y'}$$

$$\frac{\partial(\ )}{\partial t} = \frac{\partial(\ )}{\partial t'} - c_x \frac{\partial(\ )}{\partial x'} - c_y \frac{\partial(\ )}{\partial y'}$$

or

$$\nabla(\ ) = \nabla'(\ )$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \vec{c}\cdot\nabla'$$

1) $\nabla\cdot\vec{u} = 0 \quad \longrightarrow \quad \nabla'\cdot(\vec{u}' + \vec{c}) = 0 \quad \longrightarrow \quad \nabla'\cdot\vec{u}' + \overset{0}{\cancel{\nabla'\cdot\vec{c}}} = 0$

$$\Longrightarrow \quad \nabla'\cdot\vec{u}' = 0 \quad \therefore \ \underline{\text{invariant}}$$

2) $\dfrac{\partial\vec{u}}{\partial t} = 0 \quad \longrightarrow \quad \dfrac{\partial(\vec{u}' + \vec{c})}{\partial t'} - \vec{c}\cdot\nabla'(\vec{u}' + \vec{c}) = 0$

$$\frac{\partial\vec{u}'}{\partial t'} = \vec{c}\cdot\nabla'\vec{u}' \quad \therefore \ \underline{\text{not invariant}}$$

3) $\dfrac{D\vec{u}}{Dt} = 0 \quad \longrightarrow \quad \dfrac{\partial(\vec{u}' + \vec{c})}{\partial t'} + (\vec{u}' + \vec{c})\cdot\nabla'(\vec{u}' + \vec{c}) = 0$

$$\frac{\partial\vec{u}'}{\partial t'} - \cancel{\vec{c}\cdot\nabla'\vec{u}'} + \vec{u}'\cdot\nabla'\vec{u}' + \cancel{\vec{c}\cdot\nabla'\vec{u}'} = 0$$

$$\Longrightarrow \quad \frac{\partial\vec{u}'}{\partial t'} + \vec{u}'\cdot\nabla'\vec{u}' = \frac{D\vec{u}'}{Dt'} = 0 \quad \therefore \ \underline{\text{invariant}}$$

4) $\dfrac{\partial^2 \rho}{\partial t^2} - c^2\nabla^2\rho = 0 \quad \longrightarrow \quad \dfrac{\partial^2\rho}{\partial t'^2} - 2\vec{c}\cdot\nabla'\left(\dfrac{\partial\rho}{\partial t'}\right) + \vec{c}\cdot\nabla'(\vec{c}\cdot\nabla'\rho) - c^2\nabla'^2\rho = 0$

$$\frac{\partial^2(\ )}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial(\ )}{\partial t'} - \vec{c}\cdot\nabla'(\ )\right) \qquad \therefore \ \underline{\text{not invariant}}$$

$$= \frac{\partial^2(\ )}{\partial t'^2} - 2\vec{c}\cdot\nabla'\left(\frac{\partial(\ )}{\partial t'}\right) + \vec{c}\cdot\nabla'(\vec{c}\cdot\nabla'(\ ))$$