Similarity Rules

Consider the linearized equation for the perturbation equation $\Phi_1(x, y)$ in plane, steady flow:

$$(1-M_\infty^2)\frac{\partial^2\Phi_1}{\partial x^2}+\frac{\partial^2\Phi_1}{\partial y^2}=0$$

Now, consider two separate flows with their mach numbers M_{∞_1} and M_{∞_2}.

$$0 < M_{\infty_1} < 1$$
$$0 < M_{\infty_2} < 1$$
$$M_{\infty_1} \neq M_{\infty_2}$$

Also:

$$V_{\infty_1} = \text{Free stream velocity}$$
$$c = \text{Airfoil chord}$$
$$\mathrm{T}_1 = \text{Thickness ratio} = t_1/c$$
$$y = t_1 f_1\left(\frac{x}{c}\right) = \mathrm{T}_1 c f_1\left(\frac{x}{c}\right)$$
$$y = \text{Airfoil shape/boundary}$$
$$\text{or}$$
$$\frac{y}{c} = \mathrm{T}_1 f_1\left(\frac{x}{c}\right)$$

The boundary condition on the airfoil surface may be written

$$\left(\frac{\partial\Phi_1}{\partial y}\right)_{y=0} = U_{\infty_1}\left(\frac{dy}{dx}\right)_{Body} = U_{\infty_1}\mathrm{T}_1 f_1'\left(\frac{x}{c}\right)$$

From lecture notes, the linearized pressure coefficient may be written

$$C_{p_1} = -\frac{2}{U_{\infty_1}}\left(\frac{\partial\Phi_1}{\partial x}\right)_{y=0}$$

Now consider a second flow:

$$\Phi_2 = \Phi_2(\xi, \eta)$$

Assume:

$$\Phi_1(x, y) = A\frac{U_{\infty 1}}{U_{\infty_2}}\Phi_2(\xi, \eta)$$
$$\Phi_1(x, y) = A\frac{U_{\infty_1}}{U_{\infty_2}}\Phi_2(x, \left(\frac{1-M_{\infty_1}^2}{1-M_{\infty_2}^2}\right)^{1/2} y)$$

Where

A = constant, to be determined

And

$$\xi = x$$
$$2 = \left(\frac{1-M_{\infty_1}^2}{1-M_{\infty_2}^2}\right)^{1/2} y$$

Hence, equation governing $\Phi_2(\xi, \eta)$ becomes

$$(1 - M_{\infty_2}^2)\frac{\partial^2 \Phi_2}{\partial \xi^2} + \frac{\partial^2 \Phi_2}{\partial \eta^2} = 0$$

Recall:

$\Phi_1(x, y)$ is a solution corresponding to M_{∞_1}
$\Phi_2(\xi, \eta)$ is a solution corresponding to M_{∞_2}

The boundary condition along the airfoil profile may be written:

$$\left(\frac{\partial \Phi_1}{\partial x}\right)_{y=0} = A\frac{U_{\infty_1}}{U_{\infty_2}}\left(\frac{1-M_{\infty_1}^2}{1-M_{\infty_2}^2}\right)^{1/2}\left(\frac{\partial \Phi_2}{\partial \eta}\right)_{\eta=0}$$
$$= U_{\infty_1} \mathrm{T}_1 f_1'\left(\frac{x}{c}\right)$$

Now let:

$$\mathrm{T}_1 = A\sqrt{\frac{1 - M_{\infty_1}^2}{1 - M_{\infty_2}^2}}\mathrm{T}_2$$

And let $f_1 = f_2$ (f is the same in both flows) which means both airfoils are of the same family.

The boundary condition along the airfoil in (ξ, η):

$$\left(\frac{\partial \Phi_2}{\partial \eta}\right)_{\eta=0} = U_{\infty_2} \mathrm{T}_2 f_2'\left(\frac{\xi}{c}\right)$$

The pressure coefficient may be written

$$C_{p_1} = -\frac{2}{U_{\infty_1}}\left(\frac{\partial \Phi_1}{\partial x}\right)_{y=0} = -\frac{2}{U_{\infty_2}} A\left(\frac{\partial \Phi_2}{\partial \xi}\right)_{\eta=0}$$

Likewise:

$$C_{p_2} = -\frac{2}{U_{\infty_2}}\left(\frac{\partial \Phi_2}{\partial \xi}\right)_{\eta=0}$$

Therefore:

$$C_{p_1} = A C_{p_2}$$

$$A = \frac{A_1}{A_2}$$

Two airfoils of the same family of shapes characterized by the thickness ratios T_1 and T_2 have pressure distributions given by coefficients C_{p_1} and C_{p_2}. If the mach numbers of the flows are M_{∞_1} and M_{∞_2}, respectively, then $C_{p_1} = C_{p_2}$ provided:

$$\mathrm{T}_1 = A\sqrt{\frac{1 - M_{\infty_1}^2}{1 - M_{\infty_2}^2}}\mathrm{T}_2$$

$$A = \frac{A_1}{A_2}$$

Or, formally:

$$\frac{C_p}{A} = fn\left(\frac{\mathrm{T}}{A\sqrt{1 - M_{\infty}^2}}\right)$$

Recall A is a constant.

	A	C_p
(1)	1	$fn\left(\frac{\tau}{\sqrt{1-M_\infty^2}}\right)$
(2)	$\frac{1}{\sqrt{1-M_\infty^2}}$	$\frac{1}{\sqrt{1-M_\infty^2}}\, fn(\tau)$
(3)	τ	$\tau\, fn\left(\sqrt{1-M_\infty^2}\right)$
(4)	$\frac{1}{1-M_\infty^2}$	$\frac{1}{1-M_\infty^2}\, fn\left(\tau\sqrt{1-M_\infty^2}\right)$

(1), (2), (3) — PRANDTL-GLAUERT RULE

(4) — GÖTHERT RULE

	A	C_p	COMMENTS
(1)	1	$fn\ \frac{\tau}{\sqrt{1-M_\infty^2}}$	C_p INVARIANT WITH M_∞ IF $\tau/(1-M_\infty^2)^{1/2}$ = CONSTANT.
(2)	$1/(1-M_\infty^2)^{-1/2}$	$(1-M_\infty^2)^{-1/2} fn(\tau)$	FOR A GIVEN MEMBER OF THE FAMILY OF SHAPES, C_p INCREASES WITH M_∞ AS $(1-M_\infty^2)^{-1/2}$.
(3)	τ	$\tau fn(\sqrt{1-M_\infty^2})$	C_p IS PROPORTIONAL TO τ FOR A FIXED VALUE OF M_∞.
	$(1-M_\infty^2)^{-1}$	$(1-M_\infty^2)^{-1} fn(\tau\sqrt{1-M_\infty^2})$	C_p INCREASES WITH MACH NUMBER AS $(1-M_\infty^2)^{-1}$ IF τ INCREASES AS $(1-M_\infty^2)^{-1/2}$.