Singular Perturbation Methods
Formation of Shock Waves

Plane waves (small amplitude) in the absence of dissipation propagate without change in shape:

$$\frac{\partial^2 u}{\partial t^2} = a_1^2 \frac{\partial^2 u}{\partial x^2} \tag{0.1}$$

$$u(x,t) = F(x - a_1 t) + G(x + a_1 t) \tag{0.2}$$

u = particle velocity
a_1 = local speed of sound

For finite amplitude plane waves, with dissipation

$$c = a_1 \pm \frac{\gamma + 1}{2} u \tag{0.3}$$

$$c = a_1 \left\{ 1 \pm \frac{\gamma+1}{\gamma-1} \left[\left(\frac{\rho}{\rho_1} \right)^{\frac{\gamma-1}{2}} - 1 \right] \right\} \tag{0.4}$$

c = wave speed

Regions of higher condensation, $\frac{\rho}{\rho_1} > 1$, overtake those of lower condensation.

- Produces steeping effect
- Non-linear convective terms <≡> diffusive terms
- Wave becomes "stationary"

Two time scales:

(A) Viscous diffusive terms balance steep gradients generated by piston initially

(B) Non-linear convective terms balance viscous diffusive terms

Model:

Continuum flow formulation (Navier-Stokes)
ϵ = piston mach number, $\epsilon \ll 1$
Boundary conditions for large time: matching principle of inner and outer expansions
Gas is viscous and heat conducting

Non-dimensionalization:

$$\mu^* = \epsilon\sqrt{RT_0}\mu \tag{0.5}$$

$$\rho^* = \rho_0^*(1+\epsilon\rho) \tag{0.6}$$

$$p^* = p_0^*(1+\epsilon p) \tag{0.7}$$

$$T^* = T_0^*(1+\epsilon T) \tag{0.8}$$

$$\mu^* = \mu_0^*(1+\epsilon\mu) \tag{0.9}$$

$$x^* = \left[\frac{\mu_0^*}{\rho_0^*\sqrt{RT_0^*}}\right]x \tag{0.10}$$

$$t^* = \left[\frac{\mu_0^*}{\rho_0^* RT_0^*}\right]t \tag{0.11}$$

$()^*$ — dimensional variable
R — gas constant
$()_0$ — undisturbed value

Navier-Stokes Equations

$$\rho_t + \mu_x + \epsilon(\rho\mu)_x = 0 \tag{0.12}$$

$$\mu_t + p_x - \mu_{xx} + \epsilon\Big[\rho\mu_t + \mu\mu_x - (\mu\mu_x)_x\Big] + \epsilon^2\rho\mu\mu_x = 0 \tag{0.13}$$

$$T_t+(\gamma-1)\mu_x-\frac{\gamma}{\nabla}T_{xx}+\epsilon\Big[\rho T_t+\mu T_x+(\gamma-1)p\mu_x-(\gamma-1)\mu_x^2-\frac{\gamma}{\nabla}(\mu T_x)_x\Big]+\epsilon^2\Big[\rho\mu T_x+(\gamma-1)\mu\mu_x^2\Big] = 0 \tag{0.14}$$

$$p = \rho + T + \epsilon\rho T \tag{0.15}$$

ρ = specific heat ratio
∇ = Prandtl number

Initial conditions

$$\mu = \rho = \mathrm{p} = \mathrm{T} = 0;\ x > 0, t = 0$$

Boundary conditions (at piston)

$$\mu = 1, T_x = 0;\ x = \epsilon t, t > 0$$

At infinity, damping conditions

$$\mu, \rho, T \to 0, t > 0, x \to \infty$$

Expansion:

$$\mu^o = \mu_0^o + \epsilon\mu_1^o + \epsilon^2\mu_2^o + \dots \tag{0.16}$$

Linearized solutions, $\epsilon \to 0$ (small times) ("outer region")

$$\rho_t^o + \mu_x^o = 0 \tag{0.17}$$

$$\mu_t^o + p_x^o - \mu_{xx}^o = 0 \tag{0.18}$$

$$T_t^o + (\gamma-1)\mu_x^o - \frac{\gamma}{\delta}T_{xx}^o = 0 \tag{0.19}$$

$$p^o = \rho^o + T^o \tag{0.20}$$

Initial and boundary conditions

$$\mu^o = \rho^o = T^o = 0, x > 0, t = 0$$

$$\mu^o = 1, T^o_x = 0, x = 0, t > 0$$

$$\mu^o, \rho^o, T^o \to 0; t > 0, x \to \infty$$

Using Laplace transforms:

$$\overline{\mu^o}(x,s) = \int_0^\infty e^{-st}\mu^o(x,t)dt \quad (0.21)$$

$$\mu^o(x,t) \sim \frac{1}{2}\text{erfc}\left[(x-\sqrt{\gamma}t)/\sqrt{2\beta t}\right] + o(t^{-\frac{1}{2}}) \quad (0.22)$$

$$\rho^o(x,t) \sim \frac{1}{\sqrt{\gamma}}\mu^o(x,t) + o(t^{-\frac{1}{2}}) \quad (0.23)$$

$$T^o(x,t) \sim \frac{(\gamma-1)}{\sqrt{\gamma}}\mu^o(x,t) + o(t^{-\frac{1}{2}}) \quad (0.24)$$

$$\beta \equiv 1 + \frac{\gamma-1}{\nabla} \quad (0.25)$$

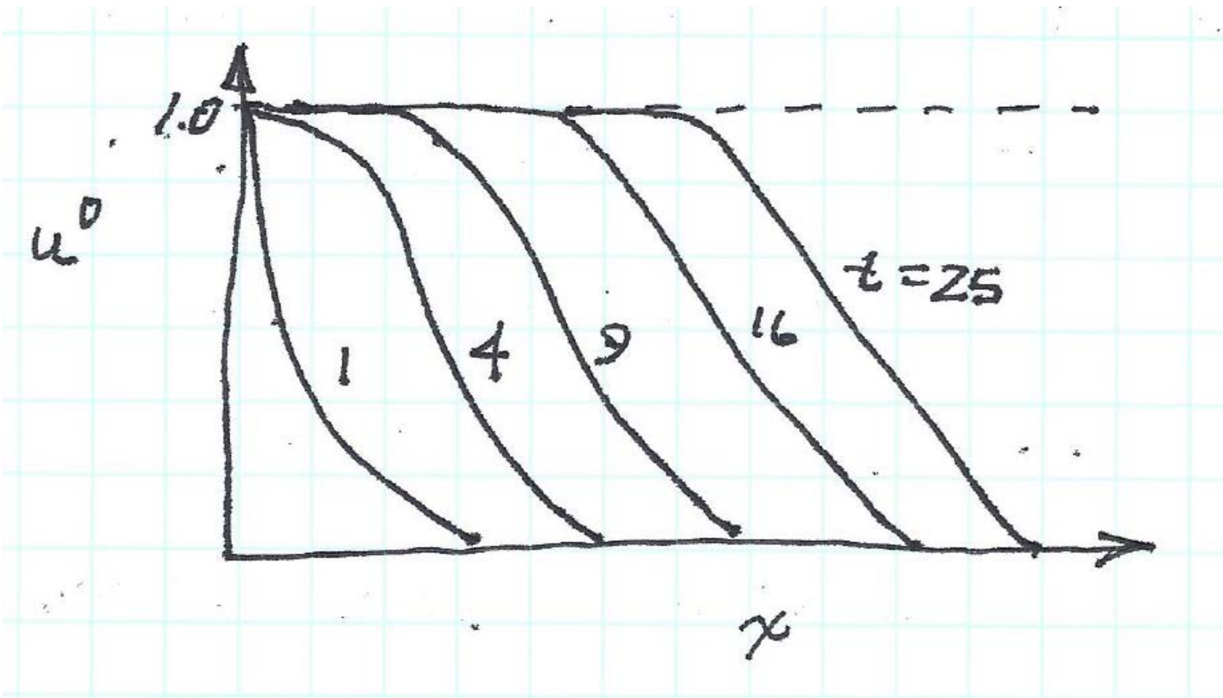
Transformed (x,t)

$$X = (x-\sqrt{\gamma}t)/\sqrt{2\beta} \quad (0.26)$$

$$\mu^o(x,t) \sim \frac{1}{2}\text{erfc}\left(\frac{X}{\sqrt{t}}\right) + o(t^{-\frac{1}{2}}) \quad (0.27)$$

$$\rho^o(x,t) \sim \frac{1}{\sqrt{\gamma}}\mu^o(x,t) + o(t^{-\frac{1}{2}}) \quad (0.28)$$

$$T^o(x,t) \sim \frac{\gamma-1}{\sqrt{\gamma}}\mu^o(x,t) + o(t^{-\frac{1}{2}}) \quad (0.29)$$



SOLUTION AT LARGE TIMES ("INNER REGION")

Re-scale t:

$$\tau = \epsilon^2 t \tag{0.30}$$

(Linearized, outer solution breaks down when $\sqrt{t} = o\left(\frac{1}{\epsilon}\right)$ or $t = o\left(\frac{1}{\epsilon^2}\right)$)
Now require a shock thickness, on the inner scale, to be order unity:

$$\xi = \epsilon(x - \sqrt{\gamma}t) = \epsilon\sqrt{2\beta}X \tag{0.31}$$

Expansion

$$\mu^i = \mu_0^i + \epsilon\mu_1^i + \epsilon^2\mu_2^i + \ldots \tag{0.32}$$

Substitute into Navier-Stokes equations, and obtain Burgers' equation:

$$\mu_\tau^i + \frac{1}{2}(\gamma+1)\mu^i\mu_\xi^i = \frac{1}{2}\beta\mu_{\xi\xi}^i \tag{0.33}$$

Boundary conditions (matching principle):

$$(\mu^i)^o = (\mu^o)^i \tag{0.34}$$

Initial conditions

$$\mu^i(\xi,0) = 0, \xi > 0 \tag{0.35}$$

$$\mu^i(\xi,0) = 1, \xi < 0 \tag{0.36}$$

Thus on the inner scale (large times) we have an initial value problem.

Consider the transformation

$$\mu^i = -\frac{2\beta}{(\gamma+1)}\frac{\psi_\xi}{\psi} \tag{0.37}$$

Burgers' equation becomes:

$$\psi_\tau = \frac{1}{2}\beta\psi_{\xi\xi} \tag{0.38}$$

(Heat conduction equation)

$$\psi(\xi,0) = \exp(-\frac{(\gamma+1)}{2\beta}\xi), \xi < 0 \tag{0.39}$$

$$\psi(\xi,0) = 1, \xi > 0 \tag{0.40}$$

Match inner and outer solutions using the asymptotic matching principle (not the limit matching principle).

Composite solution, μ^c

$$\mu^c = \mu^i + \mu^o - (\mu^i)^o \tag{0.41}$$

$$\mu^c = \mu^i\mu^o/(\mu_o)^i \tag{0.42}$$

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