Conceptual Description of Wave Drag

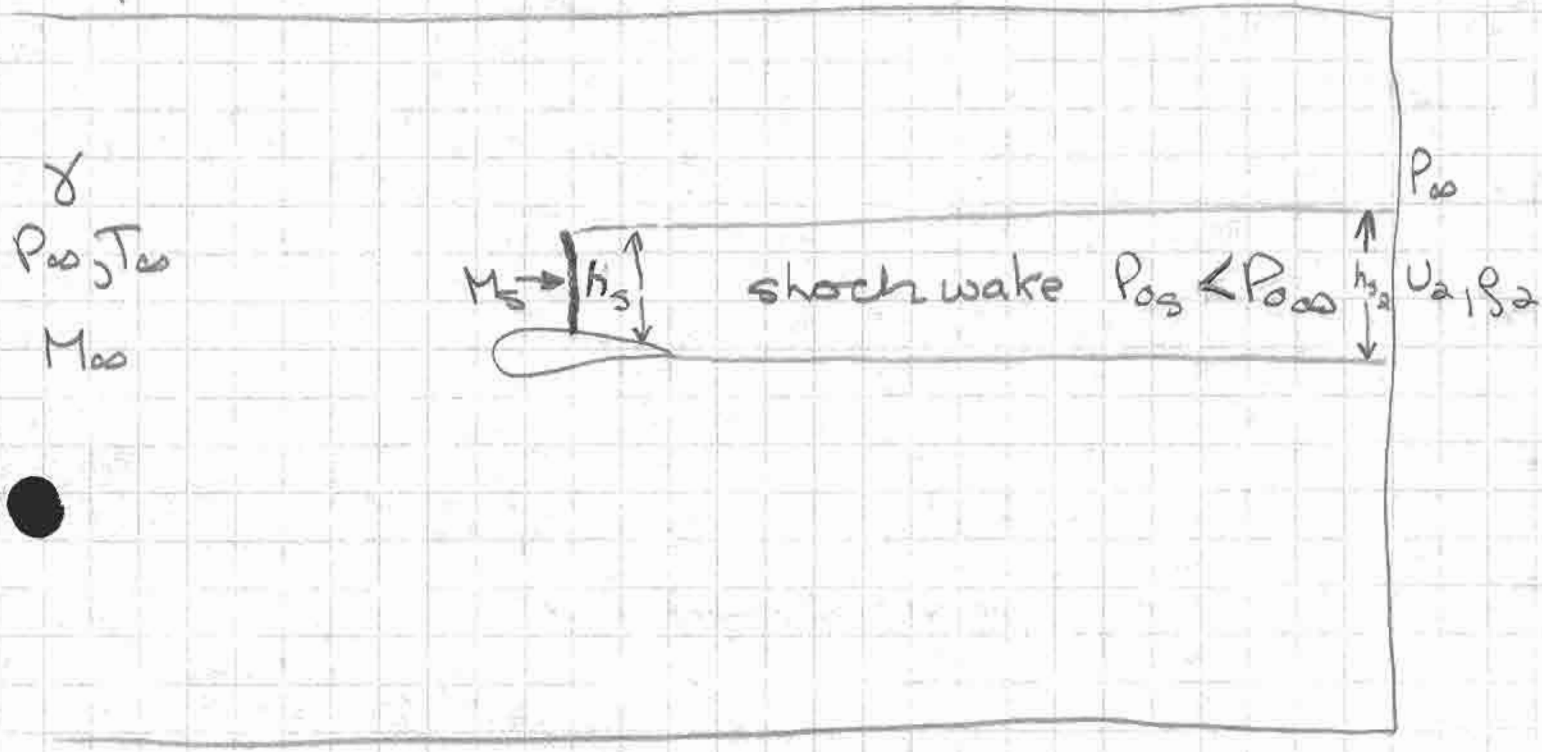
In these notes, the description of wave drag is given in terms of the total pressure decrease which occurs at a shock. To be concrete, consider a transonic airfoil with a shock wave on its upper surface:

where M_s is the Mach number upstream of the shock. Note: while we will use M_s as if it were a single value, in fact it varies from its largest at the airfoil surface to $M_s = 1$ at the tip of the shock. In short, think of M_s as an average Mach number upstream of the shock.

In section 2.6 of Anderson, the 2-D drag is shown to be equal to (assuming viscous effects are small away from the airfoil):

$$D' = \int_{-\infty}^{+\infty} \rho_2 u_2 (u_\infty - u_2)\, dy$$

where ρ_2 & u_2 are the density and velocity downstream of the airfoil where the static pressure has returned to p_∞:



Because a shock decreases the total pressure such that $p_{0s} < p_{0\infty}$, then the Mach number in the wake will be less that M_∞ because

$$p_0 = p\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}$$

and $p = p_\infty$. Thus, higher p_0 means higher M for fixed p. Similarly, since $M_2 < M_\infty$ then $u_2 < u_\infty$. Thus, the loss of total pressure leads directly to drag.

We note that behind the shock wave, the total pressure does not change in a steady, inviscid flow. Thus, by knowing the strength of the shock, the wave drag can be calculated using:

* $D' = \int_{-\infty}^{+\infty} \rho_2 u_2 (U_\infty - u_2)\, dy = \int_{0}^{h_{s_2}} \rho_2 u_2 (U_\infty - u_2)\, dy$
* Conservation of mass to relate h_s to h_{s_2}.
* Adiabatic flow ($\Rightarrow T_0 = \text{constant}$)
* P_0 does not change downstream of shock
  $\Rightarrow$ Knowing P_{0s} gives M_{s_2} from
  $$P_{0s} = P_\infty \left(1 + \frac{\gamma-1}{2} M_{s_2}^2\right)^{\frac{\gamma}{\gamma-1}}$$
* Flow returns to x-direction

To get a little insight, we will linearize assuming the total pressure loss at the shock is small:

$$\frac{P_{0\infty} - P_{0s}}{P_{0\infty}} << 1 \quad \text{where} \quad P_{0\infty} = P_\infty \left(1 + \frac{\gamma-1}{2} M_\infty^2\right)^{\frac{\gamma}{\gamma-1}}$$

Define the following quantities:

$$\Delta P_0 = P_{0s} - P_{0\infty} \qquad \Delta \rho = \rho_{s_2} - \rho_\infty \qquad \Delta a = a_{s_2} - a_\infty$$

$$\Delta U = U_{s_2} - U_\infty \qquad \Delta M = M_{s_2} - M_\infty$$

Then, the drag becomes:

$$D' = -\int_0^{h_{s_2}} (\rho_\infty + \Delta\rho)(u_\infty + \Delta u)\,\Delta u\,dy$$

$\Rightarrow$ Ignoring higher-order terms:

$$\boxed{D' \cong -\rho_\infty u_\infty \int_0^{h_{s_2}} \Delta u\,dy}$$

Now, relate changes in u to changes in M and a:

$$u = Ma$$

$$\Rightarrow \boxed{\Delta u = a\Delta M + M\Delta a} \quad (*)$$

Then, adiabatic flow gives constant total temperature:

$$T\left(1 + \frac{\gamma-1}{2}M^2\right) = T_0 = \text{const.}$$

Multiplying by γR gives:

$$a^2 + \frac{\gamma-1}{2}u^2 = \text{constant}$$

Thus, changes in a & u are related by:

$$a\Delta a + \frac{\gamma-1}{2}u\Delta u = 0$$

$$\Rightarrow \boxed{\Delta a = -\frac{\gamma-1}{2}\frac{u}{a}\Delta u}$$

Plugging this into $(*)$ gives:

$$\Delta U = a\Delta M + M\left(-\frac{\gamma-1}{2} M \Delta U\right)$$

$$\left(1+\frac{\gamma-1}{2}M^2\right)\Delta U = a\Delta M$$

$$\Rightarrow \boxed{\Delta U = \frac{a}{1+\frac{\gamma-1}{2}M^2}\,\Delta M}$$

In our case, $a = a_\infty$ & $M = M_\infty$ gives the drag as:

$$\boxed{D' \cong -\frac{\rho_\infty U_\infty a_\infty}{1+\frac{\gamma-1}{2}M_\infty^2}\int_0^{h_{S_2}} \Delta M \, dy}$$

Next, for a total pressure perturbation ΔP_0, the change in the Mach number can be found by linearizing:

$$M^2 = \frac{2}{\gamma-1}\left[\left(\frac{P_0}{P_\infty}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]$$

Giving:
$$\Delta M \cong \frac{2}{\gamma M_\infty}\left(1+\frac{\gamma-1}{2}M_\infty^2\right)\frac{\Delta P_0}{P_{0_\infty}}$$

$$\Rightarrow \boxed{D' \cong -\frac{2}{\gamma}\rho_\infty a_\infty^2 \int_0^{h_{S_2}} \frac{\Delta P_0}{P_{0_\infty}}\, dy}$$

Non-dimensionalizing:

$$\boxed{c_{d_w} = \frac{D'}{\frac{1}{2}\rho_\infty U_\infty^2 c} \cong \frac{4}{\gamma}\frac{1}{M_\infty^2}\int_0^{h_{S_2}/c} \frac{\Delta P_0}{P_{0_\infty}}\, d\left(\frac{y}{c}\right)}$$

A final approximation can be made by relating ΔP_0 to the upstream Mach number at the shock, M_s:

$$\frac{\Delta P_0}{P_{0_\infty}} \cong -\frac{2}{3}\frac{\gamma}{(\gamma+1)^2}\left(M_s^2-1\right)^3$$

$$\Rightarrow \boxed{c_{d_w} \cong \frac{8}{3}\frac{\gamma}{(\gamma+1)^2}\frac{(M_s^2-1)^3}{M_\infty^2}\frac{h_{s_2}}{c}}$$