Compressible Equations

Conservation of mass

$$\frac{d}{dt}\iiint_{volume}\rho dv+\iint_{surface}\rho\vec{u}\bullet\vec{n}dS=0$$

$$\frac{\partial\rho}{\partial t}+\nabla\bullet(\rho\vec{u})=0$$

$$\frac{D\rho}{Dt}+\rho\nabla\bullet\vec{u}=0$$

$$\nabla\bullet\vec{u}=0,\quad (incompressible)$$

Conservation of Momentum

$$\frac{d}{dt}\iiint_{volume}\rho u dv+\iint_{surface}\rho u\vec{u}\bullet\vec{n}dS=-\iint_{surface}p\vec{n}\bullet\vec{i}dS+\iint_{surface}\vec{\tau}\bullet\vec{i}dS$$

$Note:$

$$\vec{\tau}\,ds=\left(\tau_{xx}dx+\tau_{yx}dy+\tau_{zx}dz\right)\vec{i}+\left(\tau_{xy}dx+\tau_{yy}dy+\tau_{zy}dz\right)\vec{j}+\left(\tau_{xz}dx+\tau_{yz}dy+\tau_{zz}dz\right)\vec{k}$$

$$\frac{\partial(\rho u)}{\partial t}+\nabla\bullet(\rho u\vec{u})=-\frac{\partial p}{\partial x}+\underbrace{\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}}_{\text{Net viscous force in } x}$$

$$\rho\frac{Du}{Dt}=-\frac{\partial p}{\partial x}+\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}$$

Recall:

$$\tau_{ij}=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\delta_{ij}\lambda\nabla\bullet\vec{u}$$

$$\mu=\mu(T),\qquad \lambda=-\tfrac{2}{3}\mu$$

Incompressible Equations in Cartsian Coordinates

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,\ \text{i.e. }\nabla\bullet\vec{u}=0$$

$$\frac{Du}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\nabla^2 u$$

$$\frac{Dv}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\nabla^2 v$$

$$\frac{Dw}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w$$

Where

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$$

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

$$\nu\equiv\frac{\mu}{\rho},\ \text{kinematic viscosity}$$

Incompressible Equations for Cylindrical Coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(u_\theta\right)+\frac{\partial}{\partial x}\left(u_x\right)=0$$

$$\frac{Du_r}{Dt}-\frac{1}{r}u_\theta^2=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right)$$

$$\frac{Du_\theta}{Dt}+\frac{u_ru_\theta}{r}=-\frac{1}{\rho r}\frac{\partial p}{\partial\theta}+\nu\left(\nabla^2u_\theta+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2}\right)$$

$$\frac{Du_x}{Dt}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\nabla^2u_x\right)$$

Where

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}+u_z\frac{\partial}{\partial z}$$

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$$

$$\nu\equiv\frac{\mu}{\rho},\ \text{kinematic viscosity}$$