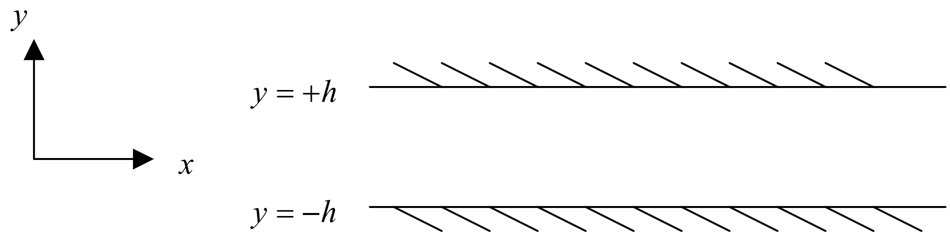
Poiseuille Flow Through a Duct in 2-D



Assumptions:

- Velocity is independent of $x, \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$
- Incompressible flow
- Constant viscosity, μ
- Steady
- Pressure gradient along length of pipe is non-zero, i.e. $\frac{\partial p}{\partial x} \neq 0$

Boundary conditions:

- No slip: $\begin{cases} u(y = \pm h) = 0 \\ v(y = \pm h) = 0 \end{cases} \Leftarrow$ walls are not moving

To be clear, we now will take the compressible, unsteady form of the N-S equations and carefully derive the solution:

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

But $\frac{\partial \rho}{\partial t} = 0$ because flow is steady and incompressible. Also, since $\rho =$ constant, then $\nabla \cdot (\rho \vec{V}) = \rho \nabla \cdot \vec{V}$

$$\Rightarrow \nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Finally, $\frac{\partial u}{\partial x} = 0$ because of assumption #1 $\rightarrow$ long pipe.

$$\Rightarrow \frac{\partial v}{\partial y} = 0$$

Now, integrate this:

$$v = \text{constant} = C$$

Apply boundary conditions: $v(\pm h) = 0 \Rightarrow \boxed{v(y) = 0}$

We expect this but it is good to see the math confirm it.

Now, let's look at $y-$momentum.

Conservation of $y-$momentum:

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$$

$$\rho \frac{\partial v}{\partial t} + \vec{V} \cdot \nabla v = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$$

$$\rho \underbrace{\cancel{\frac{\partial v}{\partial t}}}_{\substack{=0 \\ \text{steady}}} + u \underbrace{\cancel{\frac{\partial v}{\partial x}}}_{=0} + \underbrace{\cancel{v}}_{=0} \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y}$$

Now, what about $\tau_{xy} \,\&\, \tau_{yy}$

$$\tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \underbrace{\cancel{\frac{\partial v}{\partial x}}}_{=0} \right) \rightarrow \tau_{xy} = \mu \frac{\partial u}{\partial y}$$

$$\tau_{yy} = 2\mu \underbrace{\cancel{\frac{\partial v}{\partial y}}}_{v=0} + \lambda \underbrace{\cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)}}_{\nabla \cdot \vec{V} = 0} \rightarrow \tau_{yy} = 0$$

So $y-$momentum becomes:

$$0 = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial y} \right)$$

But $\frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial y} \right) = 0$ because $\frac{\partial u}{\partial x} = 0$ & $\mu = \text{constant}$

$$\Rightarrow \qquad 0 = -\frac{\partial p}{\partial y} \Rightarrow \underbrace{p(x,y,t)}_{\frac{\partial p}{\partial t}=0 \text{ (steady)}} = p(x)$$

Conservation of $x-$momentum :

$$\rho \frac{Du}{Dt} = -\underbrace{\frac{dp}{dx}}_{\substack{p=p(x)\\ \text{only}}} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}$$

$$\rho\left(\underbrace{\frac{\partial u}{\partial t}}_{\substack{=0\\ \text{steady}}} + u\underbrace{\frac{\partial u}{\partial x}}_{=0} + \underbrace{v}_{=0}\frac{\partial u}{\partial y}\right) = -\frac{dp}{dx} + \frac{\partial}{\partial x}\left(2\mu\underbrace{\frac{\partial u}{\partial x}}_{=0}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$

$$0 = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)$$

Now, we just need to solve this…

$$\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) = \frac{dp}{dx}$$

$\mu =$ const & $u = u(y)$ so,

$$\underbrace{\frac{d^2u}{dy^2}}_{\text{only} f(y)} = \underbrace{\frac{1}{\mu}\frac{dp}{dx}}_{\text{only} f(x)} \Rightarrow \text{must be constant}$$

$\Rightarrow \qquad \frac{dp}{dx} =$ const. $\Rightarrow$ pressure can only be a linear function of x!

Now, integrating twice in y gives:

$$u(y) = \frac{1}{2\mu}\frac{dp}{dx}y^2 + C_1 y + C_0$$

Finally, apply BC's:

$$u(\pm h) = 0$$

$$u(+h) = \frac{1}{2\mu}\frac{dp}{dx}h^2 + C_1 h + C_0 = 0$$

$$u(-h) = \frac{1}{2\mu}\frac{dp}{dx}h^2 - C_1 h + C_0 = 0$$

Solve for $C_0 \& C_1$ gives:

$$C_0 = -\frac{1}{2\mu}\frac{dp}{dx}h^2$$

$$C_1 = 0$$

$$\Rightarrow \quad \boxed{u(y) = \frac{-1}{2\mu h^2}\frac{dp}{dx}\left[1 - \left(\frac{y}{h}\right)^2\right]}$$