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16.07 Dynamics
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Lecture L28 - 3D Rigid Body Dynamics: Equations of Motion; Euler's Equations

3D Rigid Body Dynamics: Euler's Equations

We now turn to the task of deriving the general equations of motion for a three-dimensional rigid body. These equations are referred to as Euler's equations. The governing equations are those of conservation of linear momentum $\boldsymbol{L} = M\boldsymbol{v}_G$ and angular momentum, $\boldsymbol{H} = [I]\boldsymbol{\omega}$, where we have written the moment of inertia in matrix form to remind us that in general the direction of the angular momentum is not in the direction of the rotation vector $\boldsymbol{\omega}$. Conservation of linear momentum requires

$$\dot{\boldsymbol{L}} = \boldsymbol{F} \tag{1}$$

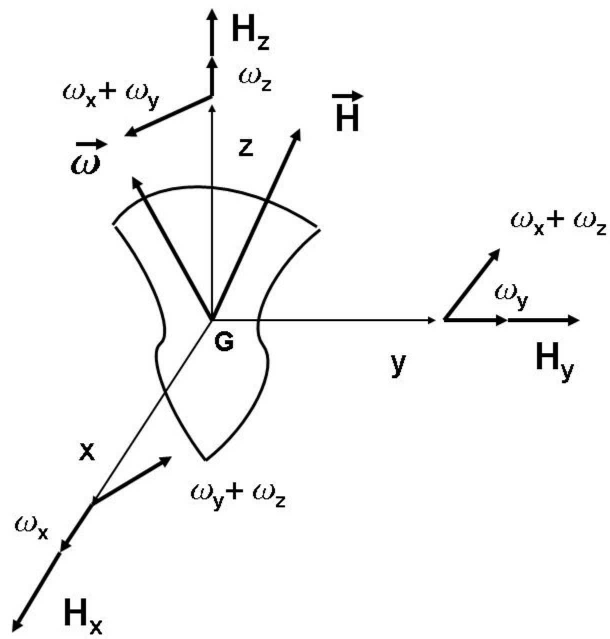
Conservation of angular momentum, about a fixed point O, requires

$$\dot{\boldsymbol{H}}_0 = \boldsymbol{M} \tag{2}$$

or about the center of mass G

$$\dot{\boldsymbol{H}}_G = \boldsymbol{M}_G \tag{3}$$

In our previous application of these equations, we specified the motion and used the equations to specify what moments would be required to produce the prescribed motion. In this more general formulation, we allow the body to execute free motions, possibly under the action of external moments. We consider the general motion of a body about its center of mass, first examining this in a inertial reference frame.



At an instant of time, we can calculate the angular momentum of the body as $\boldsymbol{H} = [I]\boldsymbol{\omega}$. One possible method to obtain the moments and the motion of the body is to perform our analysis in this inertial coordinate system. We would of course align our coordinate system initially with the principal axes of the body. We could then write

$$\boldsymbol{M}_G = \dot{\boldsymbol{H}}_G = d/dt([I]\boldsymbol{\omega}) = [\dot{I}]\boldsymbol{\omega} + [I]\dot{\boldsymbol{\omega}} \tag{4}$$

This would be a appropriate approach but the difficulty is keeping track of $[\dot{I}]$ in the inertial coordinate system. The initial inertial axis, even if principal axis, will not remain principal axis, and the inertia "seen" in this coordinate system will vary with time. So unless we are considering the motion of a sphere, for which all axis are principal and the inertia tensor is constant about all axis, we cannot get very far with this approach.

Body-Fixed Axis

We formulate the governing equations of motion in an axis system fixed to the body, paying the price for keeping track of the motion of the body in order to have the inertia tensor remain independent of time in our reference frame. Given our earlier discussion of terms added to the description of motion in a rotating and accelerating coordinate system, it may seem surprising that we can do this easily, but the statement of conservation of angular moment about the center of mass, or about a fixed point of instantaneous rotation holds if we include the changes in angular momentum arising from Coriolis Theorem. The general body is shown in the figure. We fix the x, y, z axis to the body and instantaneously align them with x, y, z. Referring to the figure, we see the components of $\boldsymbol{\omega}$,—- ω_1, ω_2 and ω_3—- and the components of the angular moment vector $\boldsymbol{H}$, which in general is not aligned with the angular velocity vector. We also see the vectors $\omega_2\mathbf{j}+\omega_3\mathbf{k}$ applied to the x axis, with corresponding components of $\boldsymbol{\omega}$ sketched at the y and z axes. At this instant of time, the change in $\boldsymbol{H}$ will be due to actual time rate of change $\dot{\boldsymbol{H}}$ plus the effect of the instantaneous rotation of the x axis due to $\boldsymbol{\omega}$: the change in $\boldsymbol{H}$ due to the instantaneous rotation of the coordinate system is

$$\dot{\boldsymbol{H}} = [I]\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{H} \tag{5}$$

Equating the change in angular momentum to the external moments, we have the statement of conservation of angular momentum in body-fixed axes.

$$M_x = \dot{H}_x - H_y\omega_z + H_z\omega_y \tag{6}$$

$$M_y = \dot{H}_y - H_z\omega_x + H_x\omega_z \tag{7}$$

$$M_z = \dot{H}_z - H_x\omega_y + H_y\omega_x \tag{8}$$

This particular form of the equations of motion is valid for any set of body-fixed axes. If the axes chosen are principal axes, then we may express the conservation of angular momentum in terms of moments of inertia about the principle axes,

$$M_x = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z \tag{9}$$

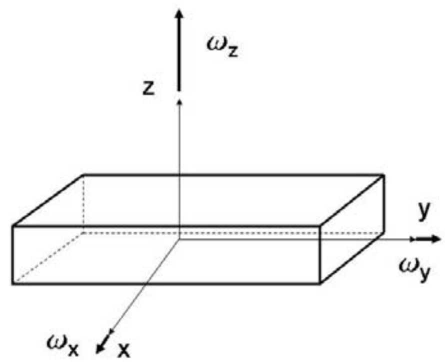
$$M_y = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x \tag{10}$$

$$M_z = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y \tag{11}$$

These equations are called Euler's equations. They provide several serious challenges to obtaining the general solution for the motion of a three-dimensional rigid body. First, they are non-linear (containing products of the unknown ω's). This means that elementary solutions cannot be combined to provide the solution for a more complex problem. But a more fundamental difficulty, is that ***we do not know the location of the axis system, x, y and z***. Recall that since the axes are fixed to the body, we are committed to follow the body as it rotates in order to use these equations to obtain a solution. Thus, we must develop a method to follow the changes in axis location as the body rotates. Before turning to this problem, we examine a situation where we know the location of the axis, at least approximately.

Stability of Free Motion about a Principal Axis

Consider a body rotating about the z axis—-a principal axis—- with angular velocity ω_z. Without loss of generality, we may consider this to be a rectangular block. We take advantage of the fact that we know to a good approximation the axis of rotation, at least initially. We examine the question of stability to small perturbations, rotations about the x and y axis of magnitude ω_x and ω_y where $\omega_x << \omega_z$ and $\omega_y << \omega_z$. The moments of inertia about the x, y and z axes are I_{xx}, I_{yy} and I_{zz}; we say nothing about the magnitudes of these inertias at this point.



Assume a small impulsive moment that initiates a small rotation about the x and y axes and thereafter the motion proceeds with no applied external moments. For this case, Euler's equations become

$$0 = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z \tag{12}$$

$$0 = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x \tag{13}$$

$$0 = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y \tag{14}$$

The z equation contains the product of two small terms in contrast to the other two equations where the size of the terms is comparable–as far as we can tell. We therefore note that since $I_{zz}\dot{\omega}_z = (I_{xx} - I_{yy})\omega_x\omega_y$ with both ω_x and ω_y being small quantities, we may take ω_z as constant, equal to some ω. We now differentiate both equations with time, and substitute to obtain an equation for ω_x. (We could do this as well for ω_y with the same result.)

$$I_{xx}\ddot{\omega}_x - \frac{(I_{yy} - (I_{zz})(I_{zz} - I_{xx})}{I_{yy}}\omega^2\omega_x = 0 \tag{15}$$

or

$$\ddot{\omega}_x - \mathbf{A}\omega_x = 0 \tag{16}$$

The solution to this differential equation for $\omega_x(t)$ is

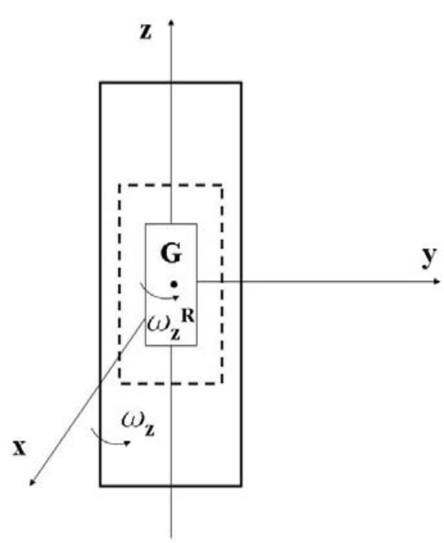
$$\omega_x(t) = Be^{\sqrt{\mathbf{A}t}} + Ce^{-\sqrt{\mathbf{A}t}} \tag{17}$$

The stability of the motion is determined by the sign of $\mathbf{A}$. If $\mathbf{A}$ is positive, an exponential divergence will result, and the initial small perturbation in ω_x will grow without bound, as least as predicted by small perturbation analysis. If the sign of $\mathbf{A}$ is negative, oscillatory motion of ω_x (and ω_y) will result, and the motion is stable.

Examining $\mathbf{A} = \frac{(I_{yy}-I_{zz})(I_{zz}-I_{xx})}{I_{yy}}$, we see that the condition for $\mathbf{A}$ to be positive is that I_{zz} is intermediate between I_{xx} and I_{yy}. That is $I_{xx} < I_{zz} < I_{yy}$ or $I_{yy} < I_{zz} < I_{xx}$. We conclude that a body rotating about an axis where the moment of inertia is intermediate between the other two inertias, is unstable. Also, that rotation about either the largest or smallest inertias is stable. This consideration relates to stability of a rotating body as predicted from Euler's equation; we have already examined the stability of rotation about the smallest inertia axis and concluded that if energy is dissipated, that motion is unstable.

Stability of a Gyrostat

The analysis of the proceeding section can easily be extended to a more important, more useful and more complex configuration: the gyrostat satellite. A gyrostat consists of a spinning body which contains within itself another spinning body, referred to as the rotor. Gyrostat satellite are used when the external body of the satellite must spin slowly to accomplish its mission while it needs the stabilization provided by faster rotation. This is accomplish by placing a rotor inside the satellite. The angular momentum of the rotor is driven by a motor attached to the satellite; no net momentum increase occurs when an adjustment in rotor speed is made. The examination of this device is quite straightforward if the rotor principal axes are aligned with the satellite principal axes, the rotor is axisymmetric, and the center of mass of the rotor is placed at the center of mass of the satellite. Thus the rotor is constrained to move with the rotational motion of the satellite and principal axes remain principal.



The moment of inertia of the rotor about its principal axes is

$$[I^R] = \begin{pmatrix} I_{xx}^R & 0 & 0 \\ 0 & I_{xx}^R & 0 \\ 0 & 0 & I_{zz}^R \end{pmatrix} \tag{18}$$

while the moment of inertia of the satellite platform is

$$[I^P] = \begin{pmatrix} I_{xx}^P & 0 & 0 \\ 0 & I_{xx}^P & 0 \\ 0 & 0 & I_{zz}^P \end{pmatrix} \tag{19}$$

Because of the geometric constraints on the system, ω_x and ω_y are equal for both platform and rotor while ω_z differ for platform, ω_z^P and rotor, ω_z^R. Since we are in a rotating coordinate system with respect to the platform so the I_{xx} and I_{yy} are constant, ω_z^R is the *relative* rotation velocity between platform and rotor. Therefore the total angular momentum is

$$[H] = \begin{pmatrix} I_{xx}^P & 0 & 0 \\ 0 & I_{yy}^P & 0 \\ 0 & 0 & I_{zz}^P \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z^P \end{pmatrix} + \begin{pmatrix} I_{xx}^P & 0 & 0 \\ 0 & I_{yy}^P & 0 \\ 0 & 0 & I_{zz}^P \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z^R \end{pmatrix} \tag{20}$$

Because of the constraints on $\boldsymbol{\omega}$, we can write the angular momentum as

$$[H] = \begin{pmatrix} I_{xx}^P + I_{xx}^R & 0 & 0 \\ 0 & I_{yy}^P + I_{xx}^R & 0 \\ 0 & 0 & I_{zz}^P + I_{zz}^R \omega_z^R / \omega_z^P \end{pmatrix} \begin{pmatrix} \omega_x^P \\ \omega_y^P \\ \omega_z^P \end{pmatrix} \tag{21}$$

where ω has been identified as ω^P.

This is a remarkable result. It indicates that the effect of the rotation on the angular momentum of the gyrostat can be incorporated by a modification of the moment of inertia about the z axis. We will call this the "effective" moment of inertia,

$$I_{zz-eff} = I_{zz}^P + I_{zz}^R \omega_z^R / \omega_z^P \tag{22}$$

where, ω_z^R is the relative rotation velocity between platform and rotor.

With this result, and the identification of the "effective" moment of inertia in z, the previous analysis of the stability of rotation of a body with unequal moments of inertia goes through with the replacement of I_{zz} by I_{zz-eff}.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition 7.9

J. B. Marion, S. T. Thornton, *Classical Dynamics of Particles and Systems*, 11.8

W.T. Thompson, *Introduction to Space Dynamics*, Chapter 5

F. P. J. Rimrott, *Introductory Attitude Dynamics*, Chapter 11

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