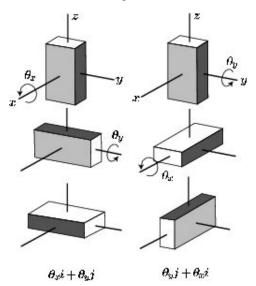# Lecture L25 - 3D Rigid Body Kinematics

In this lecture, we consider the motion of a 3D rigid body. We shall see that in the general three-dimensional case, the angular velocity of the body can change in magnitude as well as in direction, and, as a consequence, the motion is considerably more complicated than that in two dimensions.

Rotation About a Fixed Point

We consider first the simplified situation in which the 3D body moves in such a way that there is always a point, O, which is fixed. It is clear that, in this case, the path of any point in the rigid body which is at a distance r from O will be on a sphere of radius r that is centered at O. We point out that the fixed point O is not necessarily a point in rigid body (the second example in this notes illustrates this point).

Euler's theorem states that the general displacement of a rigid body, with one fixed point is a rotation about some axis. This means that any two rotations of arbitrary magnitude about different axes can always be combined into a single rotation about some axis.

At first sight, it seems that we should be able to express a rotation as a vector which has a direction along the axis of rotation and a magnitude that is equal to the angle of rotation. Unfortunately, if we consider two such rotation vectors, $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, not only would the combined rotation $\boldsymbol{\theta}$ be different from $\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2$, but in general $\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 \neq \boldsymbol{\theta}_2 + \boldsymbol{\theta}_1$. This situation is illustrated in the figure below, in which we consider a 3D rigid body undergoing two 90° rotations about the x and y axis.

$$\theta_x i + \theta_y j \qquad \theta_y j + \theta_x i$$

It is clear that the result of applying the rotation in x first and then in y is different from the result obtained by rotating first in y and then in x. Therefore, it is clear that *finite rotations* cannot be treated as vectors, since they do not satisfy simple vector operations such as the parallelogram vector addition law.

This result can also be understood by considering the rotation of axes by a coordinate transformation. Consider a transformation $(x') = [T_1](x)$, and a subsequent coordinate transformation $(x'') = [T_2](x')$. The (x'') coordinate obtained by the transformation $(x'') = [T_1][T_2](x)$ will not be the same as the coordinates obtained by the transformation $(x'') = [T_2][T_1](x)$, in other words, order matters.

Angular Velocity About a Fixed Point

On the other hand, if we consider *infinitesimal rotations* only, it is not difficult to verify that they do indeed behave as vectors. This is illustrated in the figure below, which considers the effect of two combined infinitesimal rotations, $d\boldsymbol{\theta}_1$ and $d\boldsymbol{\theta}_2$, on point A.
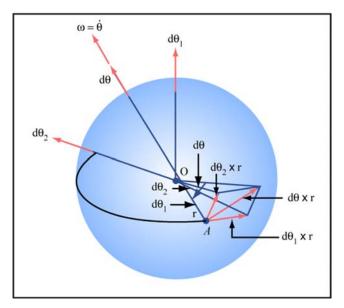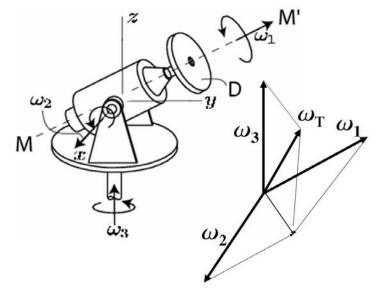

Image by MIT OpenCourseWare.

(figure reproduced from J.L. Meriam and K.L. Kraige, Dynamics, 5th edition, Wiley)

As a result of $d\boldsymbol{\theta}_1$, point A has a displacement $d\boldsymbol{\theta}_1 \times \boldsymbol{r}$, and, as a result of $d\boldsymbol{\theta}_2$, point A has a displacement $d\boldsymbol{\theta}_2 \times \boldsymbol{r}$. The total displacement of point A can then be obtained as $d\boldsymbol{\theta} \times \boldsymbol{r}$, where $d\boldsymbol{\theta} = d\boldsymbol{\theta}_1 + d\boldsymbol{\theta}_2$. Therefore, it follows that angular velocities $\boldsymbol{\omega}_1 = \dot{\boldsymbol{\theta}}_1$ and $\boldsymbol{\omega}_2 = \dot{\boldsymbol{\theta}}_2$ can be added vectorially to give $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. This means that if at any instant the body is rotating about a given axis with angular velocity $\boldsymbol{\omega}_1$ and at the same time this axis is rotating about another axis with angular velocity $\boldsymbol{\omega}_2$, the total angular velocity of the body will be simply $\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. Therefore, although the finite rotations of a body about an axis are

not vectors, the infinitesimal rotations are vectors. The angular velocity is thus a vector and for a complex configuration, the various components can ba vectorially added to obtain the total angular velocity. Consider the complex rotating configuration shown below. We want to determine the angular velocity of the disc D.

First, we note that the disc is rotating with angular velocity $\boldsymbol{\omega}_1$ about the axis MM'. In turn, this axis is rotating with angular velocity $\boldsymbol{\omega}_2$ about the horizontal axis, which is at this instant aligned with the x axis. At the same time, the whole assembly is rotating about the z axis with angular velocity $\boldsymbol{\omega}_3$. Therefore, the total angular velocity of the disc is vector sum of the individual angular velocity vectors. The resultant vector is shown in the figure: $\boldsymbol{\omega}_{total} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3$. Expressed in the fixed x, y, z system for which the configuration is instantaneously aligned as shown, we have
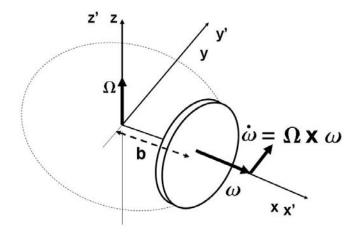
$$\boldsymbol{\omega} = \omega_2 \, \boldsymbol{i} + \omega_1 \cos \phi \, \boldsymbol{j} + (\omega_1 \sin \phi + \omega_3) \, \boldsymbol{k} \ .$$

Here, ϕ is the angle between MM' and the y axis.

Angular Acceleration

In order to apply the principle of conservation of angular momentum, we need a general expression for the angular acceleration of the various components of a complex rotating configuration. We first examine this problem with respect to inertial coordinate system, x, y, z. In order to apply conservation of angular momentum, we must develop a formula for the time rate of change of angular momentum, $\dot{BH} = \frac{d}{dt}(I_G\boldsymbol{\omega}) = I_G\frac{d}{dt}\boldsymbol{\omega} + \boldsymbol{\omega}\frac{d}{dt}I_G$. We will later consider under what conditions we need to consider $\boldsymbol{\omega}\frac{d}{dt}I_G$.

For now, we concentrate on determining $\frac{d}{dt}\boldsymbol{\omega}$. We first consider a common situation in the study of rotating 3D objects, the rotating wheel with angular velocity $\boldsymbol{\omega}$ attached to a central hub which rotates with angular velocity $\boldsymbol{\Omega}$, shown in the figure. The total angular velocity is the vector sum of $\boldsymbol{\Omega}$ and $\boldsymbol{\omega}$.
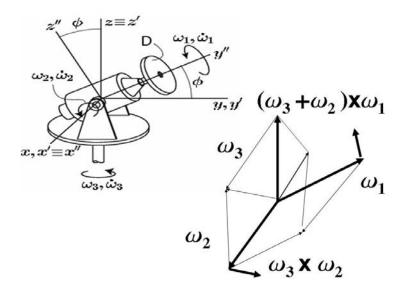
In this example, we take $\boldsymbol{\omega}$ constant in magnitude (but not direction) and $\boldsymbol{\Omega}$ as constant. It is clear that the rotation $\boldsymbol{\Omega}$ will rotate the vector $\boldsymbol{\omega}$, changing its direction. The magnitude of $\dot{\boldsymbol{\omega}}$ is $\Omega\omega$, the direction is normal to $\boldsymbol{\omega}$; by Coriolis theorem, the result is $\dot{\boldsymbol{\omega}} = \boldsymbol{\Omega} \times \boldsymbol{\omega}$. It is interesting to note that this result is independent of the distance b between the wheel and the axis of rotation for $\boldsymbol{\Omega}$. This is a consequence of our earlier observation that in a rigid body rotating with angular velocity $\boldsymbol{\omega}$, every point rotates with angular velocity $\boldsymbol{\omega}$.

Example   **Multiple Observers**

In this example we illustrate a more systematic procedure for calculating the angular velocities and accelerations when several reference frames are involved. We want to determine the angular acceleration of the disc D as a function of the angular velocities and accelerations given in the diagram. The angle of $\boldsymbol{\omega}_1$ with the horizontal is ϕ.

The angular velocity vector for each component is the vector sum of the individual angular velocities of the components:ω_3, $\omega_{2_{total}} = \omega_3 + \omega_2$; $\omega_{1_{total}} = \omega_1 + \omega_2 + \omega_3$. The angular accelerations of the various components are worked through individually. Consider ω_3. Since it is the "primary" rotor, the rotation ω_2 and ω_1 do not affect its motion. Therefore the angular acceleration is due solely to $\dot{\omega}_3$. Consider ω_2. This angular velocity vector will change with time both due to $\dot{\omega}_2$ as well as $\omega_3 \times \omega_2$. And finally, for $\dot{\omega}_1$: this angular velocity vector will change with time both due to $\dot{\omega}_1$ as well as $(\omega_3 + \omega_2) \times \omega_1$.

We now resolve these vectors into appropriate coordinate systems. We consider three sets of axes. Axes xyz are fixed. Axes $x'y'z'$ rotate with angular velocity $\omega_3 \boldsymbol{k}$ with respect to xyz. Axes $x''y''z''$ rotate with angular velocity $\omega_2 \boldsymbol{i}'$ with respect to $x'y'z'$. Finally, the disc rotates with angular velocity $\omega_1 \boldsymbol{j}''$ with respect to the axes $x''y''z''$.

The angular velocity of the disc with respect to the fixed axes will be simply

$$\boldsymbol{\Omega} = \omega_2 \boldsymbol{i}' + \omega_1 \boldsymbol{j}'' + \omega_3 \boldsymbol{k} \ . \tag{1}$$

At the instant considered, $\boldsymbol{i}' = \boldsymbol{i}$ and $\boldsymbol{j}'' = \cos\phi \boldsymbol{j} + \sin\phi \boldsymbol{k}$. Therefore, we can also write,

$$\boldsymbol{\Omega} = \omega_2 \boldsymbol{i} + \omega_1 \cos\phi \boldsymbol{j} + (\omega_1 \sin\phi + \omega_3)\boldsymbol{k} \ .$$

In order to calculate the angular acceleration of the disc with respect to the fixed axes xyz, we start from (1) and write,

$$\left(\frac{d\boldsymbol{\Omega}}{dt}\right)_{xyz} = \left(\frac{d}{dt}(\omega_2 \boldsymbol{i}' + \omega_1 \boldsymbol{j}'' + \omega_3 \boldsymbol{k})\right)_{xyz} = \left(\frac{d}{dt}(\omega_2 \boldsymbol{i}' + \omega_1 \boldsymbol{j}'')\right)_{xyz} + \dot{\omega}_3 \boldsymbol{k} \ .$$

Here, we have used the fact that $\boldsymbol{k}$ does not change with respect to the xyz axes and therefore only the magnitude of ω_3 changes. In order to calculate the time derivative of $\omega_2 \boldsymbol{i}' + \omega_1 \boldsymbol{j}''$ with respect to the inertial reference frame, we apply Coriolis' theorem. Since $x'y'z'$ rotates with angular velocity $\omega_3 \boldsymbol{k}$ with respect to

xyz, we write,

$$\left(\frac{d}{dt}(\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'')\right)_{xyz} = \left(\frac{d}{dt}(\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'')\right)_{x'y'z'} + \omega_3\boldsymbol{k} \times (\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'') \ .$$

In the $x'y'z'$ frame, $\boldsymbol{i}'$ does not change direction. Therefore, we can write

$$\left(\frac{d}{dt}(\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'')\right)_{xyz} = \dot{\omega}_2\boldsymbol{i}' + \left(\frac{d}{dt}(\omega_1\boldsymbol{j}'')\right)_{x'y'z'} + \omega_3\boldsymbol{k} \times (\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'') \ .$$

In order to evaluate the derivative of $\omega_1\boldsymbol{j}''$ with respect to the $x'y'z'$ frame we make use again of Coriolis' theorem, and write

$$\left(\frac{d}{dt}(\omega_1\boldsymbol{j}'')\right)_{x'y'z'} = \left(\frac{d}{dt}(\omega_1\boldsymbol{j}'')\right)_{x''y''z''} + \omega_2\boldsymbol{i}' \times \omega_1\boldsymbol{j}'' = \dot{\omega}_1\boldsymbol{j}'' + \omega_2\boldsymbol{i}' \times \omega_1\boldsymbol{j}'' \ .$$

Here, we have used the fact that $x''y''z''$ rotates with angular velocity $\omega_2\boldsymbol{i}'$ with respect to $x'y'z'$, and the derivative of $\boldsymbol{j}''$ in the $x''y''z''$ reference frame is zero. Putting it all together, we have,

$$
\begin{aligned}
\left(\frac{d\boldsymbol{\Omega}}{dt}\right)_{xyz} &= \dot{\omega}_2\boldsymbol{i}' + \dot{\omega}_1\boldsymbol{j}'' + \omega_2\boldsymbol{i}' \times \omega_1\boldsymbol{j}'' + \omega_3\boldsymbol{k} \times (\omega_2\boldsymbol{i}' + \omega_1\boldsymbol{j}'') + \dot{\omega}_3\boldsymbol{k} \\
&= \dot{\omega}_2\boldsymbol{i}' + \dot{\omega}_1\boldsymbol{j}'' + \omega_1\omega_2\boldsymbol{k}'' + \omega_2\omega_3\boldsymbol{j} - \omega_1\omega_3\cos\phi\,\boldsymbol{i} + \dot{\omega}_3\boldsymbol{k} \\
&= (\dot{\omega}_2 - \omega_1\omega_3\cos\phi)\,\boldsymbol{i} + (\dot{\omega}_1\cos\phi + \omega_2\omega_3 - \omega_1\omega_2\sin\phi)\,\boldsymbol{j} + (\dot{\omega}_3 + \dot{\omega}_1\sin\phi + \omega_1\omega_2\cos\phi)\,\boldsymbol{k}.
\end{aligned}
$$

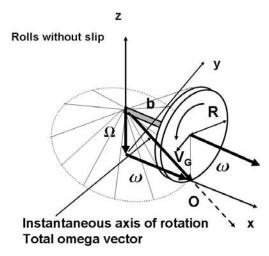## Instantaneous Axis of Rotation

In two dimensions, we introduced the concept of instantaneous center of rotation. For a rotating body with one fixed point, we can extend this concept to an ***instantaneous axis of rotation.***

Consider a vertical disc rotating with a constant angular velocity $\boldsymbol{\omega}$, rolling without slip with an angular velocity about the vertical axis with angular velocity $\boldsymbol{\Omega}$. The no-slip condition requires that the velocity of the center of mass $V_G = -\omega R$. The disk rolling around a circle of radius b will have an angular velocity $\Omega = \omega R/b$. We note that just as in two-dimensions, point O is a fixed point, an instantaneous center of rotation. That is, at that instant, the motion of the disc is such that the distance from any point in the disc to the point O remains constant. The total instantaneous angular velocity is the vector sum of $\boldsymbol{\Omega} + \boldsymbol{\omega}$. The resultant vector is sketched and as would be expected, it is parallel to a line from the center of the hub to the point C.

Once the instantaneous angular velocity, $\boldsymbol{\omega}_T = \boldsymbol{\Omega} + \boldsymbol{\omega}$, has been determined, the velocity of any point in the rigid body is simply
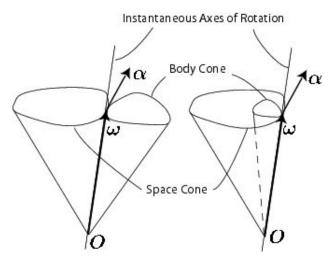
$$\boldsymbol{v} = \boldsymbol{\omega}_T \times \boldsymbol{r} \ , \tag{2}$$

where $\boldsymbol{r}$ is the position vector of the point considered with respect to the fixed point O. It follows that for any point which is on the line passing through O and parallel to $\boldsymbol{\omega}_T$, the velocity will be zero. This line is therefore the *Instantaneous Axis of Rotation.*

Rolls without slip

Instantaneous axis of rotation
Total omega vector

We can now define two space curves, which are instantaneously tangent to the instantaneous axis of rotation. These curves are useful for more complex problems in the general rotation of a body about a fixed point. This simple example gives us a useful visualization of these space curves. The first is called the **body cone**: it is the *locus of points made by the instantaneous axis of rotation as the body traces its motion*. As can be seen, for this case, it is a cone of radius equal to the radius of the disc, extending to the hub at the z axis. (The term **body cone** is a bit of a misnomer; the body doesn't have to fit entirely inside the body cone.) The second space curve is called the **space cone**. It is defined as the *locus of points traced in space by the instantaneous axis of rotation*. At the instantaneous position of the body, these two curves are both tangent to the instantaneous axis of rotation. In this case, the space cone is the cone bounded by the curve traced by the instantaneous contact point and the fixed point, the center of the hub. The body curve and the space curve for more general three-dimensional motions are shown in the figure.

For a general motion, as the direction of the instantaneous axis of rotation (or the line passing through O parallel to $\boldsymbol{\omega}$) changes in space, the locus of points defined by the axis generates a fixed *Space Cone*. If the change in this axis is viewed with respect to the rotating body, the locus of the axis generates a *Body Cone*.

At any given instant, these two cones are tangent along the instantaneous axis of rotation. When the body is in motion, the body cone appears to roll either on the inside or the outside of the fixed space cone. This situation is illustrated in the figure below.

The acceleration of any point in the rigid body is obtained by taking the derivative of expression 2. Thus,

$$\boldsymbol{a} = \dot{\boldsymbol{\omega}}_T \times \boldsymbol{r} + \boldsymbol{\omega}_T \times \dot{\boldsymbol{r}} = \boldsymbol{\alpha} \times \boldsymbol{r} + \boldsymbol{\omega}_T \times (\boldsymbol{\omega}_T \times \boldsymbol{r}) \ . \tag{3}$$

Here, $\boldsymbol{\alpha}$ is the angular acceleration vector and is locally tangent to both the Space and the Body Cones.

General Motion

In the general case, the displacement of a rigid body is determined by a translation plus a rotation about some axis. This result is a generalization of Euler's theorem, which is sometimes known as Chasles' theorem. In practice, this means that six parameters are needed to define the position of a 3D rigid body. For instance, we could choose three coordinates to specify the position of the center of mass, two angles to define the axis of rotation and an additional angle to determine the magnitude of the rotation.

Unlike the motion about a fixed point, it is not always possible to define an instantaneous axis of rotation. Consider, for instance, a body which is rotating with angular velocity $\boldsymbol{\omega}$ and, at the same time, has a translational velocity parallel to $\boldsymbol{\omega}$. It is clear that, in this case, all the points in the body have a non-zero velocity, and therefore an instantaneous center of rotation cannot be defined.

It turns out that, in some situations, the motion of the center of mass of a 3D rigid body can be determined independent of the orientation. Consider, for instance, the motion of an orbiting satellite in free flight. In this situation, the sum of all external forces on the satellite does not depend on the satellite's attitude, and, therefore, it is possible to determine the position without knowing the attitude. In more complex situations, however, it may be necessary to solve simultaneously for both the position of the center of mass and the attitude.

The velocity, $\boldsymbol{v}_P$, and acceleration, $\boldsymbol{a}_P$, of a point, P, in the rigid body can be determined if we know the velocity, $\boldsymbol{v}_{O'}$, and acceleration, $\boldsymbol{a}_{O'}$, of a point in the rigid body, O', as well as the body's angular velocity, $\boldsymbol{\omega}$, and acceleration, $\boldsymbol{\alpha}$.

$$\boldsymbol{v}_P = \boldsymbol{v}_{O'} + \boldsymbol{\omega} \times \boldsymbol{r}'_P \tag{4}$$

$$\boldsymbol{a}_P = \boldsymbol{a}_{O'} + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}'_P + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'_P) \ . \tag{5}$$

Here, $\boldsymbol{r}'_P$ is the position vector of the point, P, relative to O'. We point out that the angular velocity and angular acceleration are the same for all the points in the rigid body.

ADDITIONAL READING

J.L. Meriam and L.G. Kraige, *Engineering Mechanics, DYNAMICS*, 5th Edition

7/1, 7/2, 7/3, 7/4, 7/5, 7/6 (review)

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