Lecture S14

The Bilateral Laplace Transform

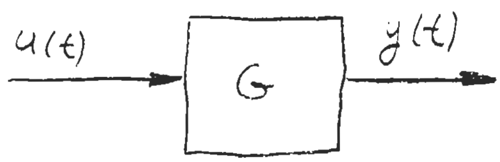
The unilateral LT has two limitations:

- It works only for <u>causal</u> systems
- It works only for "causal" inputs.

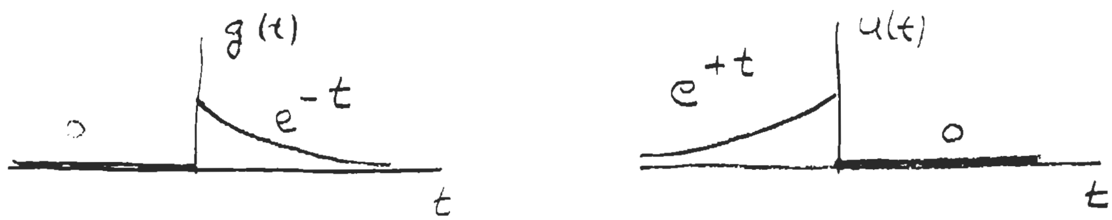
The first limitation is not unreasonable — most (but not all) systems are causal.

The second limitation is not reasonable — why should $u(t) = 0$ for $t < 0$?

<u>Example</u> For the system



$$y(t) = e^{-t}\sigma(t)$$
$$u(t) = e^{t}\sigma(-t)$$



causal noncausal

Find $y(t)$.

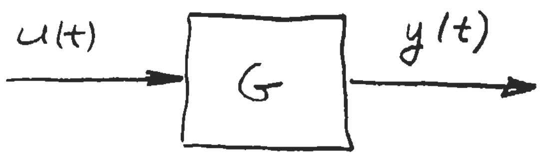
We <u>can't</u> solve using unilateral LT — $U(s)$ doesn't make sense.

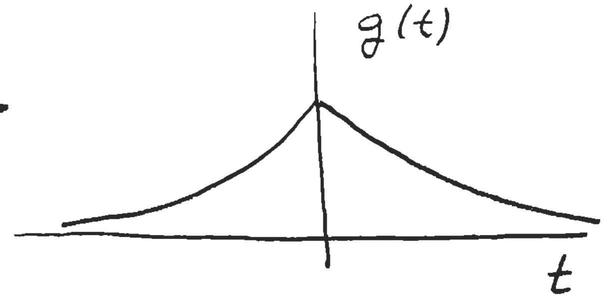
This limitation is a serious defect; but can be eliminated by working with the bilateral transform.

To motivate the bilateral LT, find ...

The Transfer Function of a Noncausal System

Example "smoothing filter"





This $g(t)$ cannot be implemented as a (causal) circuit or differential equation. It can be implemented in post-processing, say, of a noisy data set.

Suppose $u(t) = e^{st}$. What is $y(t)$? We know that, except in exceptional circumstances,

$$y(t) = G(s)e^{st}$$

where

$$G(s) = \text{transfer function}$$

This is the definition of the transfer fcn.

If the system is LTI,

$$
\begin{aligned}
y(t) &= g(t) * u(t) \\
&= u(t) * g(t) \qquad \left[\begin{array}{l}\text{convolution is} \\ \text{commutative}\end{array}\right] \\
&= \int_{-\infty}^{\infty} g(\tau)\, u(t-\tau)\, d\tau
\end{aligned}
$$

g not causal!

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} g(\tau)\, e^{s(t-\tau)}\, d\tau \\
&= e^{st} \underbrace{\int_{-\infty}^{\infty} g(\tau)\, e^{-s\tau}\, d\tau}_{G(s)}
\end{aligned}
$$

So the transfer function is the bilateral Laplace transform of the impulse response:

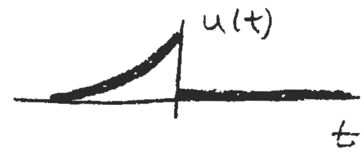
$$
\boxed{G(s) = \int_{\underline{\underline{-\infty}}}^{\infty} g(t)\, e^{-st}\, dt = \mathcal{L}[g(t)]}
$$

When $g(t)$ is causal, this reduces to the unilateral LT.

The unilateral LT is a <u>special</u> <u>case</u> of the bilateral LT.

<u>Example</u> $u(t) = e^{t}\sigma(-t) = \begin{cases} e^{t}, & t \leq 0 \\ 0, & t > 0 \end{cases}$

What is $\mathcal{L}[u(t)]$?



$$U(s) = \mathcal{L}[u(t)]$$

$$= \int_{-\infty}^{\infty} u(t) e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{t} e^{-st}\, dt = \int_{-\infty}^{0} e^{(1-s)t}\, dt$$

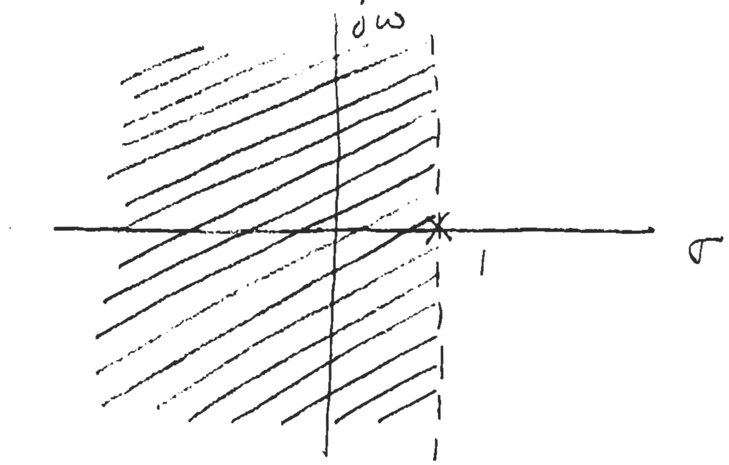
$$= \frac{1}{1-s} e^{(1-s)t} \Big|_{-\infty}^{0}$$

$$= \frac{1}{1-s}(1 - 0) = \frac{-1}{s-1}$$

$\underline{\underline{if}}$ $1-s > 0 \Rightarrow s < 1$

$$\mathcal{L}[e^{t}\sigma(-t)] = \frac{-1}{s-1}, \quad \mathrm{Re}[s] < 1$$

So region of convergence is



Note that

$$\mathcal{L}\left[e^{t}\sigma(-t)\right] = \frac{-1}{s-1}, \quad Re[s] < 1$$

$$\mathcal{L}\left[-e^{t}\sigma(t)\right] = \frac{-1}{s-1}, \quad Re[s] > 1$$

Only difference is the region of convergence! The inverse LT is unique only when the r.o.c. is specified.

In general,

$$g(t) = \underbrace{g(t)\sigma(t)}_{\text{positive time part}} + \underbrace{g(t)\sigma(-t)}_{\text{negative time part.}}$$

$$\Rightarrow \mathcal{L}[g(t)] = \mathcal{L}[g(t)\sigma(t)] + \mathcal{L}[g(t)\sigma(-t)]$$

$$= \underbrace{\int_0^\infty g(t)\, e^{-st}\, dt}_{\substack{\text{converges for} \\ Re[s] > \sigma_1}} + \underbrace{\int_0^\infty g(t)\, e^{-st}\, dt}_{\substack{\text{Converges for} \\ Re[s] < \sigma_2}}$$

more + s makes exponential decay faster

more − s makes exponential decay (to the left) faster.

So if LT converges, it will converge in a strip

$$\sigma_1 < Re[s] < \sigma_2$$